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Estimation of the drift of fractional Brownian motion

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Abstract

We consider the problem of efficient estimation for the drift of fractional
Brownian motion $B^H := (B^H_t)_{t \in [0,T]}$ with hurst parameter $H$ less than $\frac{1}{2}$. We
also construct superefficient James-Stein type estimators which dominate, under
the usual quadratic risk, the natural maximum likelihood estimator.

Key words: Fractional Brownian Motion, Stein estimate, MLE

2000 Mathematics Subject Classification: 60G15, 62G05, 62B05, 62M09.

1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \{(B^H_t, s)_{t \in [0,T]}\}$ be a $d$-dimensional
fractional Brownian motion (fBm) defined on the probability space $(\Omega, \mathcal{F}, P)$. That
is, $B^H$ is a zero mean Gaussian vector whose components are independent one-
dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for
every $i = 1, \ldots, d$ $B^{H,i}_t$ is a Gaussian process and covariance function given by

$$E(B^H_s, B^H_t) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}) , \quad s, t \in [0,T].$$

For each $i = 1, \ldots, d$, $(\mathcal{F}^i_t)_{t \in [0,T]}$ denotes the filtration generated by $\left(B^{H,i}_t\right)_{t \in [0,T]}$.

The fBm was first introduced by [3] and studied by [4]. Notice that if $H = \frac{1}{2}$, the
process $B^H_t$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let $M$ be a subspace of the Cameron-Martin space defined by

$$M = \left\{ \varphi : [0,T] \to \mathbb{R}^d; \varphi_t^i = \int_0^t \dot{\varphi}_s^i ds \text{ with } \varphi_t^i \in L^2([0,T]) \right. $$

$$\left. \text{ and } \varphi_t^i \in L^{H+\frac{1}{2}} \left( L^2([0,T]) \right), i = 1, \ldots, d \right\}.$$ 

Let $\theta = \{ (\theta_t^1, \ldots, \theta_t^d); t \in [0,T] \}$ be a function belonging to $M$. Then, Applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to $P$ under which the process $\tilde{B}^H_t$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0,T]$$

is a fBm with Hurst parameter $H$ and mean zero. In this case, we say that, under the probability $P_{\theta}$, the process $B^H_t$ is a fBm with drift $\theta$.

We consider in this paper the problem of estimating the drift $\theta$ of $B^H_t$ under the probability $P_{\theta}$, with hurst parameter $H < 1/2$. We wish to estimate $\theta$ under the usual quadratic risk, that is defined for any estimator $\delta$ of $\theta$ by

$$R(\theta, \delta) = E_{\theta} \left[ \int_0^T |\delta_t - \theta_t|^2 dt \right]$$

where $E_{\theta}$ is the expectation with respect to a probability $P_{\theta}$.

Let $X = (X^1, \ldots, X^d)$ be a normal vector with mean $\theta = (\theta^1, \ldots, \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of $\theta$ is $X$ itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by $X$. That is

$$\sigma^2 d = E \left[ \| X - \theta \|^2 \right] = \inf_{\xi \in \mathcal{S}} E \left[ \| \xi - \theta \|^2 \right],$$

where $\mathcal{S}$ is the class of unbiased estimators of $\theta$ and $\| . \|_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

[12] constructed biased superefficient estimators of $\theta$ of the form

$$\delta_{a,b}(X) = \left( 1 - \frac{b}{a + \| X \|^2} \right) X$$

for $a$ sufficiently small and $b$ sufficiently large when $d \geq 3$. [4] sharpened later this result and presented an explicit class of biased superefficient estimators of the form
\[
\left(1 - \frac{a}{\|X\|_d^2}\right) X, \text{ for } 0 < a < 2(d - 2).
\]

Recently, an infinite-dimensional extension of this result has been given by [10]. The authors constructed unbiased estimators of the drift $(\theta_t)_{t \in [0,T]}$ of a continuous Gaussian martingale $(X_t)_{t \in [0,T]}$ with quadratic variation $\sigma_t^2 dt$, where $\sigma \in L^2([0,T], dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t \in [0,T]}$ is an efficient estimator of $(\theta_t)_{t \in [0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian process of the form:
\[
X_t := \int_0^t K(t, s) dW_s, \quad t \in [0, T],
\]
where $(W_t)_{t \in [0,T]}$ is a standard Brownian motion and $K(\ldots)$ is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where $F$ is a positive superharmonic random variable and $D$ is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\hat{\theta} = B^H$ is an efficient estimator of $\theta$ under the probability $P_\theta$ with risk
\[
\mathcal{R}(\theta, B^H) = E_\theta \left[ \int_0^T \|B^H_t - \theta_t\|^2 dt \right] = \frac{T^{2H+1}}{2H+1} d.
\]
Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of $\theta$.

In Section 4, we construct a class of biased estimators of James-Stein type of the form
\[
\delta(B^H)_t = \left(1 - at^{2H} \left(\frac{r(\|B^H_t\|^2)}{\|B^H_t\|^2}\right)\right) B^H_t, \quad t \in [0, T].
\]
We give sufficient conditions on the function $r$ and on the constant $a$ in order that $\delta(B^H)$ dominates $B^H$ under the usual quadratic risk i.e.
\[
\mathcal{R}(\theta, \delta(B^H)) < \mathcal{R}(\theta, B^H) \quad \text{for all } \theta \in M.
\] 

2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.
The fractional Brownian motion $B^H$ has the following stochastic integral representation (see for instance, [1], [8])

$$B^H_i = \int_0^t K_H(t,s) dW^i_s, \quad i = 1, \ldots, d; \quad t \in [0,T] \quad (2.3)$$

where $W = (W^1, \ldots, W^d)$ denotes the $d$-dimensional Brownian motion and the kernel $K_H(t,s)$ is equal to

$$c_H(t-s)^{H-\frac{1}{2}} + c_H \left( \frac{1}{2} - H \right) \int_s^t (u-s)^{H-\frac{1}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2} - H} \right) du \quad \text{if } H \leq \frac{1}{2}$$

$$c_H(H-\frac{1}{2}) \int_s^t (u-s)^{H-\frac{3}{2}} \left( \frac{s}{u} \right)^{H-\frac{1}{2}} du \quad \text{if } H > \frac{1}{2},$$

if $s < t$ and $K_H(t,s) = 0$ if $s \geq t$. Here $c_H$ is the normalizing constant

$$c_H = \left( \frac{2H \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{\frac{1}{2}}$$

where $\Gamma$ is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [3].

The left fractional Riemann-Liouville integral of $f \in L^1((a,b))$ of order $\alpha > 0$ on $(a,b)$ is given at almost all $x \in (a,b)$ by

$$I^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy.$$ 

If $f \in L^p_a((a,b))$ with $0 < \alpha < 1$ and $p > 1$ then the left-sided Riemann-Liouville derivative of $f$ of order $\alpha$ defined by

$$D^\alpha_a f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right)$$

for almost all $x \in (a,b)$.

For $H \in (0,1)$, the integral transform

$$(K_H f)(t) = \int_0^t K_H(t,s) f(s)ds$$
is a isomorphism from $L^2([0,1])$ onto $L_0^{H+\frac{1}{2}}(L^2([0,1]))$ and its inverse operator $K_H^{-1}$ is given by

$$K_H^{-1} f = t^{H-\frac{1}{2}}D_0^H t^{H-\frac{1}{2}} f' \text{ for } H > 1/2,$$

$$K_H^{-1} f = t^{\frac{1}{2}-H}D_0^{\frac{1}{2}-H} t^{H-\frac{1}{2}}D_0^H f \text{ for } H < 1/2.$$ (2.4)

Moreover, for $H < \frac{1}{2}$, if $f$ is an absolutely continuous function then $K_H^{-1} f$ can be represented of the form (see [11])

$$K_H^{-1} f = t^{H-\frac{1}{2}}I_0^{\frac{1}{2}-H} t^{\frac{1}{2}-H} f'.$$ (2.5)

3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function $\theta = (\theta^1, \ldots, \theta^d)$ belonging to $M$. An estimator $\xi = (\xi^1, \ldots, \xi^d)$ of $\theta = (\theta^1, \ldots, \theta^d)$ is called unbiased if, for every $t \in [0,T]$ 

$$E_{\theta}(\xi^i_t) = \theta^i_t, \quad i = 1, \ldots, d$$

and it is called adapted if, for each $i = 1, \ldots, d$, $\xi^i$ is adapted to $\mathcal{F}_t$. Since for any $i = 1, \ldots, d$, the function $\theta^i$ is deterministic and 

$$\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem yields that there exists a probability measure $P_{\theta}$ absolutely continuous with respect to $P$ under which the process $\tilde{B}^H := (\tilde{B}_t^H; t \in [0,T])$ defined by

$$\tilde{B}_t^H = B_t^H - \theta_t, \quad t \in [0,T]$$ (3.7)

is a d-dimensional fBm with Hurst parameter $H$ and mean zero. Moreover the Girsanov density of $P_{\theta}$ with respect to $P$ is given by:

$$\frac{dP_{\theta}}{dP} = \exp \left[ \sum_{i=1}^d \left( \int_0^T K_H^{-1}(\theta^i)(s)dW^i_s - \frac{1}{2} \int_0^T (K_H^{-1}(\theta^i)(s))^2 ds \right) \right]$$
and
\[ \tilde{B}^H_t = \int_0^t K_H(t, s) d\tilde{W}_s \]
where \( \tilde{W} \) is a d-dimensional Brownian motion under the probability \( P_\theta \) and
\[ \tilde{W}^i_t = W^i_t - \int_0^t K^{-1}_H(\theta^i)(s) ds, \quad i = 1, \ldots, d; \quad t \in [0, T]. \]
The equation (3.7) implies that \( B^H \) is an unbiased and adapted estimator of \( \theta \) under probability \( P_\theta \). In addition, we obtain the Cramer-Rao type bound:
\[ R(H, \hat{\theta}) := R(\theta, B^H) = \int_0^T E_\theta \| \tilde{B}^H_t \|^2 dt = d \int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d. \]
The first main result of this section is given by the following proposition which asserts that \( \hat{\theta} = B^H \) is an efficient estimator of \( \theta \).

**Theorem 1** Assume that \( H < \frac{1}{2} \). If \( \xi \) is an unbiased and adapted estimator of \( \theta \), then
\[ E_\theta \int_0^T \| \xi_t - \theta_t \|^2 dt \geq R(H, \hat{\theta}). \quad (3.8) \]

**Proof:** Since \( \xi \) is unbiased, then for every \( \varphi \in M \) we have
\[ E_\varphi(\xi^j_t) = E_\varphi(\varphi^j_t), \quad j = 1, \ldots, d. \]
Let \( \varphi = \theta + \varepsilon \psi \) with \( \psi \in M \) and \( \varepsilon \in \mathbb{R} \). Then for every \( t \in [0, T] \) and \( j \in \{1, \ldots, d\} \), we have
\[ E_{\theta + \varepsilon \psi}(\xi^j_t) = E_{\theta + \varepsilon \psi}(\theta^j_t + \varepsilon \psi^j_t) = E_{\theta + \varepsilon \psi}(\theta^j_t) + \varepsilon \psi^j_t. \]
This implies that for every $j = 1, \ldots, d$

\[
\psi_j(t) = \frac{d}{d\varepsilon} E_{\theta+\varepsilon} \langle \xi_j - \theta_j \rangle
\]

\[
= E \left( \frac{d}{d\varepsilon} E_{\theta+\varepsilon} \left[ \sum_{i=1}^{d} \left( \int_0^t K_H^{-1}(\theta^i + \varepsilon \psi^i)(s) dW_s^i - \frac{1}{2} \int_0^t (K_H^{-1}(\theta^i + \varepsilon \psi^i)(s))^2 ds \right) \right] \langle \xi_j - \theta_j \rangle \right)
\]

\[
= E \left( \sum_{i=1}^{d} \left[ \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\theta^i)(s) ds \right] \langle \xi_j - \theta_j \rangle \right)
\]

\[
= E \left( \int_0^t K_H^{-1}(\psi^j)(s) d\tilde{W}_s^j \langle \xi_j - \theta_j \rangle \right)
\]

Applying Cauchy-Schwarz inequality in $L^2(\Omega, dP_\theta)$, we obtain that for every $t \in [0, T]$

\[
\|\psi_t\|^2 = \sum_{j=1}^{d} (\psi_j(t))^2 \leq \sum_{j=1}^{d} E_{\theta} \left( (\xi_j - \theta_j)^2 \right) E_{\theta} \left( \left[ \int_0^t K_H^{-1}(\psi^j)(s) d\tilde{W}_s^j \right]^2 \right)
\]

\[
= \sum_{j=1}^{d} E_{\theta} \left[ (\xi_j - \theta_j)^2 \int_0^t (K_H^{-1}(\psi^j)(s))^2 ds \right]
\]

We take for each $j = 1, \ldots, d$, $\psi_j(t) = t^{2H}$. Since $t \rightarrow t^{2H}$ is absolutely continuous function, then by (2.6), a simple calculation shows that

\[
K_H^{-1}(t^{2H}) = 2Ht^{H-\frac{1}{2}} \int_0^t (t^{-H})^{\frac{1}{2}}
\]

\[
= 2H \beta(\frac{1}{2} - H, H + \frac{1}{2}) t^{H-1/2}
\]

\[
= 2H(\Gamma(H + \frac{1}{2}) t^{H-1/2}.
\]

It is known that

\[
0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1, 2]. \quad (3.9)
\]
Combining the facts that $z\Gamma(z) = \Gamma(z + 1)$, $z > 0$, $2H \leq (H + \frac{1}{2})^2$ and (3.9), we obtain

$$dt^{2H} = \|\psi_t\|^2 \leq (\Gamma(\frac{3}{2} + H))^2 E_\theta \left(\|\xi_t - \theta_t\|^2\right) \leq E_\theta \left(\|\xi_t - \theta_t\|^2\right).$$

Hence, by an integration with respect to $dt$, we get

$$R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt.$$ 

Therefore (3.8) is satisfied.

**Corollary 1** The process $\hat{\theta} = B^H$ is a maximum likelihood estimator of $\theta$.

**Proof:** We have for every $\psi \in M$

$$\frac{d}{d\varepsilon |\varepsilon=0} \exp \left[ \sum_{i=1}^d \int_0^t K_H^{-1}(\hat{\theta}^i + \varepsilon \psi^i)(s)dW_s^i - \frac{1}{2} \int_0^t (K_H^{-1}(\hat{\theta}^i + \varepsilon \psi^i)(s))^2 ds \right] = 0.$$

Hence

$$\sum_{i=1}^d \left( \int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s) ds \right) = 0.$$

Which implies that for every $i = 1, \ldots, d$

$$E \left( \int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s) ds \right)^2 = 0.$$

Then, for each $i = 1, \ldots, d$

$$W_t^i = \int_0^t K_H^{-1}(\hat{\theta}^i)(s) ds, \quad t \in [0, T].$$

Therefore by (2.3), we obtain that $B^H = \hat{\theta}$.

**4 Superefficient James-Stein type estimators**

The aim of this section is to construct superefficient estimators of $\theta$ which dominate, under the usual quadratic risk, the natural MLE estimator $B^H$. The class of estimators considered here are of the form

$$\delta(B^H)_t = B^H_t + g(B^H_t, t), \quad t \in [0, T]$$

(4.10)
where \( g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) is a function. The problem turns to find sufficient conditions on \( g \) such that \( \mathcal{R}(\theta, \delta(B^H)) < \infty \) and the risk difference is negative, i.e.

\[
\Delta \mathcal{R}(\theta) = \mathcal{R}(\theta, \delta(B^H)) - \mathcal{R}(\theta, B^H) < 0.
\]

In the sequel we assume that the function \( g \) satisfies the following assumption:

\[(A) \quad \left\{ \begin{array}{l}
E_\theta \left[ \int_0^T |g(B^H_t, t)|_d^2 dt \right] < \infty, \\
\text{the partial derivatives } \partial_i g^j := \frac{\partial g^j}{\partial x^i}, \ i = 1, \ldots, n \text{ of } g \text{ exist.}
\end{array} \right.\]

Then \( \mathcal{R}(\theta, \delta(B^H)) < \infty \). Moreover

\[
\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \|B^H_t + g(B^H_t, t) - \theta_t\|_d^2 - \|B^H_t - \theta_t\|_d^2 dt \right]
\]

\[
= E_\theta \left[ \int_0^T \|g(B^H_t, t)\|_d^2 + 2 \sum_{i=1}^d \left( g^i(B^H_t, t)(B^H_t - \theta^i_t) \right) dt \right].
\]

In addition,

\[
E_\theta \int_0^T \sum_{i=1}^d \left( g^i(B^H_t, t)(B^H_t - \theta^i_t) \right) dt
\]

\[
= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} g^i(x^1, \ldots, x^d, t)(x^i - \theta^i_t) \right.
\]

\[\times e^{-\frac{\sum_{j=1}^d (x^j - \theta^j)^2}{2t^{2H}}} dx^1 \ldots dx^d \left) dt \right.
\]

\[
= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t) \right.
\]

\[\times e^{-\frac{\sum_{j=1}^d (x^j - \theta^j)^2}{2t^{2H}}} dx^1 \ldots dx^d \left) dt \right.
\]

\[
= \sum_{i=1}^d \int_0^T t^{2H} E_\theta \partial_i g^i(B^H_t, t) dt = E_\theta \left[ \sum_{i=1}^d \int_0^T t^{2H} \partial_i g^i(B^H_t, t) dt \right].
\]

Consequently, the risk difference equals

\[
\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( \|g(B^H_t, t)\|_d^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B^H_t, t) \right) dt \right]. \quad (4.11)
\]
We can now state the following theorem.

**Theorem 2** Let $g : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $(B_t^H + g(B_t^H, t))_{t \in [0,T]}$ to dominate $B_t^H$ under the usual quadratic risk is

$$E_\theta \left[ \int_0^T \left( \|g(B_t^H, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B_t^H, t) \right) dt \right] < 0. \tag{4.12}$$

As an application, take $g$ of the form

$$g(x, t) = at^{2H} \frac{(\|x\|^2)}{\|x\|^2}x, \tag{4.12}$$

where $a$ is a constant strictly positive and $r : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded derivable function.

The next lemma give a sufficient condition for $g$ in (4.12) to satisfies the assumption (A).

**Lemma 1** If $d \geq 3$ and $H < \frac{1}{2}$ then

$$E \left[ \int_0^T \frac{1}{\|B_t^H\|^2} dt \right] < \infty. \tag{4.13}$$

**Proof:** Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)((t, w); B_t^H(w) = 0) = 0$$

where $(dt \times P)$ is the product measure.

Using the change of variable and $d \geq 3$ we see that

$$E \int_0^T \frac{1}{\|B_t^H\|^2} dt = \int_0^T \frac{dt}{t^{2H}} \int_{\mathbb{R}^d} e^{-\frac{\|y\|^2}{2}} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,$$

where $C$ is a constant depending only on $d$. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

**Theorem 3** Assume that $d \geq 3$. If the function $r$, the constant $a$ and the parameter $H$ satisfy:

i) $0 \leq r(\cdot) \leq 1$

ii) $r(\cdot)$ is differentiable and increasing
iii) \( 0 < a \leq 2(d - 2) \) and \( H < 1/2 \), then the estimator
\[
\delta(B^H) = B^H_t - at^{2H} \frac{r(||B^H_t||^2)}{||B^H_t||^2} B^H_t, \quad t \in [0, T].
\]
dominates \( B^H \).

**Proof:** It suffices to prove that \( \Delta R(\theta) < 0 \). From (4.11) and the hypothesis i) and ii) we can write
\[
\Delta R(\theta) = a E_\theta \left[ \int_0^T t^{4H} \left( \frac{ar^2(||B^H_t||^2)}{||B^H_t||^2} - 2(d - 2)\frac{r(||B^H_t||^2)}{||B^H_t||^2} \right. \\
\left. - 4r'(||B^H_t||^2) \right) dt \right]
\]
\[
\leq a [a - 2(d - 2)] E_\theta \left[ a \int_0^T t^{4H} \frac{r(||B^H_t||^2)}{||B^H_t||^2} \right].
\]
Combining this fact with the assumption iii) yields that the risk difference is negative. Which proves the desired result.

For \( r = 1 \), we obtain a James-Stein type estimator:

**Corollary 2** Let \( d \geq 3, 0 < H < \frac{1}{2} \) and \( 0 < a \leq 2(d - 2) \). Then the estimator
\[
\left( 1 - \frac{a t^{2H}}{||B^H_t||^2} \right) B^H_t, \quad t \in [0, T]
\]
dominates \( B^H \).

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