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Estimation of the drift of fractional Brownian motion

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Abstract

We consider the problem of efficient estimation for the drift of fractional Brownian motion $B_H := (B^H_t)_{t \in [0,T]}$ with hurst parameter $H$ less than $\frac{1}{2}$. We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

Key words : Fractional Brownian Motion, Stein estimate, MLE

2000 Mathematics Subject Classification: 60G15, 62G05, 62B05, 62M09.

1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \{(B^H_{t,1}, ..., B^H_{t,d}); t \in [0,T]\}$ be a $d$-dimensional fractional Brownian motion (fBm) defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, $B^H$ is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for every $i = 1, ..., d$ $B^{H,i}$ is a Gaussian process and covariance function given by

$$E(B^{H,i}_s B^{H,i}_t) = \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H}), \quad s,t \in [0,T].$$

For each $i = 1, \ldots, d$, $(\mathcal{F}_t)_{t \in [0,T]}$ denotes the filtration generated by $(B^H_{t,i})_{t \in [0,T]}$. The fBm was first introduced by [3] and studied by [2]. Notice that if $H = \frac{1}{2}$, the
process $B^{1/2}$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let $M$ be a subspace of the Cameron-Martin space defined by

$$M = \left\{ \varphi : [0, T] \to \mathbb{R}^d; \varphi^i_t = \int_0^t \dot{\varphi}^i_s ds \text{ with } \varphi^i \in L^2([0, T]) \text{ and } \varphi^i \in L^{H+\frac{1}{2}}_2([0, T]), i = 1, \ldots, d \right\}.$$ 

Let $\theta = \{ (\theta^1_t, \ldots, \theta^d_t); t \in [0, T] \}$ be a function belonging to $M$. Then, applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to $P$ under which the process $\tilde{B}^H_t$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0, T] \quad (1.1)$$

is a fBm with Hurst parameter $H$ and mean zero. In this case, we say that, under the probability $P_{\theta}$, the process $B^H$ is a fBm with drift $\theta$.

We consider in this paper the problem of estimating the drift $\theta$ of $B^H$ under the probability $P_{\theta}$, with hurst parameter $H < 1/2$. We wish to estimate $\theta$ under the usual quadratic risk, that is defined for any estimator $\delta$ of $\theta$ by

$$R(\theta, \delta) = E_{\theta} \left[ \int_0^T |\delta_t - \theta_t|^2 dt \right]$$

where $E_{\theta}$ is the expectation with respect to a probability $P_{\theta}$.

Let $X = (X^1, \ldots, X^d)$ be a normal vector with mean $\theta = (\theta^1, \ldots, \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of $\theta$ is $X$. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by $X$. That is

$$\sigma^2 d = E \left[ \|X - \theta\|_2^2 \right] = \inf_{\xi \in S} E \left[ \|\xi - \theta\|_2^2 \right],$$

where $S$ is the class of unbiased estimators of $\theta$ and $\|.\|_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

[12] constructed biased superefficient estimators of $\theta$ of the form

$$\delta_{a,b}(X) = \left( 1 - \frac{b}{a + \|X\|_2^2} \right) X$$
for $a$ sufficiently small and $b$ sufficiently large when $d \geq 3$. They sharpened later this result and presented an explicit class of biased superefficient estimators of the form
\[
\left(1 - \frac{a}{\|X\|^2_d}\right)X, \quad 0 < a < 2(d - 2).
\]

Recently, an infinite-dimensional extension of this result has been given by [10]. The authors constructed unbiased estimators of the drift $(\theta_t)_{t \in [0,T]}$ of a continuous Gaussian martingale $(X_t)_{t \in [0,T]}$ with quadratic variation $\sigma_t^2 dt$, where $\sigma \in L^2([0,T], dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t \in [0,T]}$ is an efficient estimator of $(\theta_t)_{t \in [0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian process of the form:
\[
X_t := \int_0^t K(t,s) dW_s, \quad t \in [0,T],
\]
where $(W_t)_{t \in [0,T]}$ is a standard Brownian motion and $K(\cdot, \cdot)$ is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where $F$ is a positive superharmonic random variable and $D$ is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\hat{\theta} = B^H$ is an efficient estimator of $\theta$ under the probability $P_{\theta}$ with risk
\[
\mathcal{R}(\theta, B^H) = E_\theta \left[ \int_0^T \| B^H_t - \theta_t \|^2 dt \right] = \frac{T^{2H+1}}{2H + 1}. 
\]
Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of $\theta$.

In Section 4, we construct a class of biased estimators of James-Stein type of the form
\[
\delta(B^H)_t = \left(1 - at^{2H} \left( r(\|B^H_t\|^2) \right) \right) B^H_t, \quad t \in [0,T].
\]
We give sufficient conditions on the function $r$ and on the constant $a$ in order that $\delta(B^H)$ dominates $B^H$ under the usual quadratic risk i.e.
\[
\mathcal{R}(\theta, \delta(B^H)) < \mathcal{R}(\theta, B^H) \quad \text{for all } \theta \in M. \quad (1.2)
\]

2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.
The fractional Brownian motion $B^H$ has the following stochastic integral representation (see for instance, \cite{1}, \cite{8})

$$B^H_{t,i} = \int_0^t K_H(t,s)dW^i_s, \quad i = 1,\ldots,d; \quad t \in [0,T]$$

(2.3)

where $W = (W^1,\ldots,W^d)$ denotes the d-dimensional Brownian motion and the kernel $K_H(t,s)$ is equal to

$$c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2} - H\right) \int_s^t \left(1 - \frac{s}{u}\right)^{H-\frac{1}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} du \quad \text{if } H \leq \frac{1}{2}$$

$$c_H(H - \frac{1}{2}) \int_s^t \left(\frac{s}{u}\right)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} du \quad \text{if } H > \frac{1}{2}$$

if $s < t$ and $K_H(t,s) = 0$ if $s \geq t$. Here $c_H$ is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2-2H)}}$$

where $\Gamma$ is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in \cite{11}.

The left fractional Riemann-Liouville integral of $f \in L^1((a,b))$ of order $\alpha > 0$ on $(a,b)$ is given at almost all $x \in (a,b)$ by

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy.$$  

If $f \in L^\alpha_{a+}(L^p(a,b))$ with $0 < \alpha < 1$ and $p > 1$ then the left-sided Riemann-Liouville derivative of $f$ of order $\alpha$ defined by

$$D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy\right)$$

for almost all $x \in (a,b)$.

For $H \in (0,1)$, the integral transform

$$(K_H f)(t) = \int_0^t K_H(t,s)f(s)ds$$

4
is a isomorphism from $L^2([0,1])$ onto $L^2([0,1])$ and its inverse operator $K_H^{-1}$ is given by

$$
K_H^{-1}f = \begin{cases} 
  t^{H-1/2}D_0^{H-1/2}t_{1/2-H}'f' & \text{for } H > 1/2, \\
  t^{1/2-H}D_0^{1/2-H}t_{H-1/2}D_0^{2H}f & \text{for } H < 1/2.
\end{cases}
$$

Moreover, for $H < 1/2$, if $f$ is an absolutely continuous function then $K_H^{-1}f$ can be represented of the form (see [4])

$$
K_H^{-1}f = t^{H-1/2}D_0^{1/2-H}t_{1/2-H}'f'.
$$

\section{The maximum likelihood estimator and Cramer-Rao type bound}
We consider a function $\theta = (\theta^1, \ldots, \theta^d)$ belonging to $M$. An estimator $\xi = (\xi^1, \ldots, \xi^d)$ of $\theta = (\theta^1, \ldots, \theta^d)$ is called unbiased if, for every $t \in [0,T]

$$
E_\theta(\xi_i^t) = \theta_i \quad i = 1, \ldots, d
$$

and it is called adapted if, for each $i = 1, \ldots, d$, $\xi_i^t$ is adapted to $(\mathcal{F}_t^i)_{t \in [0,T]}$. Since for any $i = 1, \ldots, d$, the function $\theta_i$ is deterministic and

$$
\int_0^T (K_H^{-1}(\theta_i^t)')^2 ds < \infty,
$$

then Girsanov theorem yields that there exists a probability measure $P_\theta$ absolutely continuous with respect to $P$ under which the process $\tilde{B}_t^H := (\tilde{B}_t^H; t \in [0,T])$ defined by

$$
\tilde{B}_t^H = B_t^H - \theta_t, \quad t \in [0,T]
$$

is a $d$-dimensional fBm with Hurst parameter $H$ and mean zero. Moreover the Girsanov density of $P_\theta$ with respect to $P$ is given by:

$$
\frac{dP_\theta}{dP} = \exp \left[ \sum_{i=1}^d \left( \int_0^T K_H^{-1}(\theta_i^t) dW_i^t - \frac{1}{2} \int_0^T (K_H^{-1}(\theta_i^t))' ds \right) \right]
$$
and 
\[ \tilde{B}_t^H = \int_0^t K_H(t, s) d\tilde{W}_s \]
where \( \tilde{W} \) is a d-dimensional Brownian motion under the probability \( P_\theta \) and 
\[ \tilde{W}_t^i = W_t^i - \int_0^t K_H^{-1}(\theta^i)(s) ds, \quad i = 1, \ldots, d; \quad t \in [0, T]. \]

The equation (3.7) implies that \( B^H \) is an unbiased and adapted estimator of \( \theta \) under probability \( P_\theta \). In addition, we obtain the Cramer-Rao type bound:
\[ R(H, \hat{\theta}) := R(\theta, B^H) = \int_0^T E_\theta \| \tilde{B}_t^H \|^2 dt = d \int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d. \]

The first main result of this section is given by the following proposition which asserts that \( \hat{\theta} = B^H \) is an efficient estimator of \( \theta \).

**Theorem 1** Assume that \( H < \frac{1}{2} \). If \( \xi \) is an unbiased and adapted estimator of \( \theta \), then
\[ E_\theta \int_0^T \| \xi_t - \theta_t \|^2 dt \geq R(H, \hat{\theta}). \quad (3.8) \]

**Proof:** Since \( \xi \) is unbiased, then for every \( \varphi \in M \) we have 
\[ E_\varphi(\xi_t^j) = E_\varphi(\varphi_t^j), \quad j = 1, \ldots, d. \]
Let \( \varphi = \theta + \varepsilon \psi \) with \( \psi \in M \) and \( \varepsilon \in \mathbb{R} \). Then for every \( t \in [0, T] \) and \( j \in \{1, \ldots, d\} \), we have
\[ E_{\theta + \varepsilon \psi}(\xi_t^j) = E_{\theta + \varepsilon \psi}(\theta_t^j + \varepsilon \psi_t^j) = E_{\theta + \varepsilon \psi}(\theta_t^j) + \varepsilon \psi_t^j. \]
This implies that for every \( j = 1, \ldots, d \)

\[
\psi_t^j = \frac{d}{d \varepsilon_{t=0}} E_{\theta + \varepsilon \psi} (\xi_t^j - \theta_t^j)
\]

\[
= E \left( \frac{d}{d \varepsilon_{t=0}} \exp \left[ \sum_{i=1}^d \left( \int_0^t K_H^{-1}(\theta^i + \varepsilon \psi^i)(s)dW_s^i \right. \right.ight.
\]
\[
\left. \left. - \frac{1}{2} \int_0^t (K_H^{-1}(\theta^i + \varepsilon \psi^i)(s))^2 ds \right) \right] (\xi_t^j - \theta_t^j) \right)
\]

\[
= E \left( \sum_{i=1}^d \left[ \int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\theta^i)(s)ds \right] \right)
\]
\[
\times (\xi_t^j - \theta_t^j) \right)
\]

\[
= E \left( \sum_{i=1}^d \left[ \int_0^t K_H^{-1}(\psi^i)(s)d\tilde{W}_s^i \right] (\xi_t^j - \theta_t^j) \right)
\]

\[
= E \left( \left[ \int_0^t K_H^{-1}(\psi^j)(s)d\tilde{W}_s^j \right] (\xi_t^j - \theta_t^j) \right).
\]

Applying Cauchy-Schwarz inequality in \( L^2(\Omega, dP_\theta) \), we obtain that for every \( t \in [0, T] \)

\[
||\psi_t||^2 = \sum_{j=1}^d (\psi_t^j)^2 \leq \sum_{j=1}^d E_{\theta} \left( (\xi_t^j - \theta_t^j)^2 \right) E_{\theta} \left( \left[ \int_0^t K_H^{-1}(\psi^j)(s)d\tilde{W}_s^j \right]^2 \right)
\]
\[
= \sum_{j=1}^d E_{\theta} \left[ \left( (\xi_t^j - \theta_t^j)^2 \right) \int_0^t (K_H^{-1}(\psi^j)(s))^2 ds \right].
\]

We take for each \( j = 1, \ldots, d, \psi_t^j = t^{2H} \). Since \( t \to t^{2H} \) is absolutely continuous function, then by (2.6), a simple calculation shows that

\[
K_H^{-1}(t^{2H}) = 2H t^{H - \frac{1}{2}} \frac{\Gamma(\frac{1}{2} - H)}{\Gamma(\frac{1}{2} - H)} t^{H/2}
\]
\[
= 2H \beta(\frac{1}{2} - H, H + \frac{1}{2}) t^{H - 1/2}
\]
\[
= 2H (\Gamma(\frac{1}{2} + H)) t^{H - 1/2}.
\]

It is known that

\[
0 \leq \Gamma(z) \leq 1 \quad \text{for every} \quad z \in [1, 2]. \tag{3.9}
\]
Combining the facts that $z \Gamma(z) = \Gamma(z + 1)$, $z > 0$, $2H \leq \left( H + \frac{1}{2} \right)^2$ and (3.9), we obtain

$$dt^{2H} = \|\psi_t\|^2 \leq \left( \Gamma\left(\frac{3}{2} + H\right) \right)^2 E_\theta \left( \|\xi_t - \theta_t\|^2 \right) \leq E_\theta \left( \|\xi_t - \theta_t\|^2 \right).$$

Hence, by an integration with respect to $dt$, we get

$$R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt.$$

Therefore (3.8) is satisfied.

**Corollary 1** The process $\hat{\theta} = B^H$ is a maximum likelihood estimator of $\theta$.

**Proof**: We have for every $\psi \in M$

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \exp \left[ \sum_{i=1}^d \int_0^t K_H^{-1}(\hat{\psi}^i + \varepsilon \psi^i)(s) dW_s^i - \frac{1}{2} \int_0^t (K_H^{-1}(\hat{\psi}^i + \varepsilon \psi^i)(s))^2 ds \right] = 0.$$

Hence

$$\sum_{i=1}^d \left( \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\hat{\psi}^i)(s) ds \right) = 0.$$

Which implies that for every $i = 1, ..., d$

$$E \left( \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\hat{\psi}^i)(s) ds \right)^2 = 0.$$

Then, for each $i = 1, ..., d$

$$W_t^i = \int_0^t K_H^{-1}(\hat{\psi}^i)(s) ds, \quad t \in [0, T].$$

Therefore by (2.3), we obtain that $B^H = \hat{\theta}$.

**4 Superefficient James-Stein type estimators**

The aim of this section is to construct superefficient estimators of $\theta$ which dominate, under the usual quadratic risk, the natural MLE estimator $B^H$. The class of estimators considered here are of the form

$$\delta(B^H)_t = B^H_t + g(B^H_t, t), \quad t \in [0, T] \quad (4.10)$$
where $g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ is a function. The problem turns to find sufficient conditions on $g$ such that $\mathcal{R}(\theta, \delta(B^H)) < \infty$ and the risk difference is negative, i.e.

$$\Delta \mathcal{R}(\theta) = \mathcal{R}(\theta, \delta(B^H)) - \mathcal{R}(\theta, B^H) < 0.$$ 

In the sequel we assume that the function $g$ satisfies the following assumption:

$$(A) \left\{ \begin{align*}
E_\theta \left[ \int_0^T ||g(B_t^H, t)||_d^2 dt \right] < \infty, \\
\text{the partial derivatives } \partial_i g^i := \frac{\partial g^i}{\partial x^i}, \; i = 1, \ldots, n \text{ of } g \text{ exist.}
\end{align*} \right.$$ 

Then $\mathcal{R}(\theta, \delta(B^H)) < \infty$. Moreover

$$\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( ||B_t^H + g(B_t^H, t) - \theta_t||_d^2 - ||B_t^H - \theta_t||_d^2 \right) dt \right]$$

$$= E_\theta \left[ \int_0^T ||g(B_t^H, t)||_d^2 + 2 \sum_{i=1}^d \left( g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i) \right) dt \right].$$

In addition,

$$E_\theta \int_0^T \sum_{i=1}^d \left( g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i) \right) dt$$

$$= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} g^i(x^1, \ldots, x^d, t)(x^i - \theta_t^i) \right.$$

$$\times e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)^2}{2t^{2H}}} dx^1 \ldots dx^d \right) dt$$

$$= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t) \right.$$ 

$$\times e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)^2}{2t^{2H}}} dx^1 \ldots dx^d \right) dt$$

$$= \sum_{i=1}^d \int_0^T \left( t^{2H} E_\theta \partial_i g^i(B_t^H, t) \right) dt = E_\theta \left[ \sum_{i=1}^d \int_0^T \left( t^{2H} \partial_i g^i(B_t^H, t) \right) dt \right].$$

Consequently, the risk difference equals

$$\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( ||g(B_t^H, t)||_d^2 + 2 t^{2H} \sum_{i=1}^d \partial_i g^i(B_t^H, t) \right) dt \right].$$
We can now state the following theorem.

**Theorem 2** Let $g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $(B_t^H + g(B_t^H, t))_{t \in [0,T]}$ to dominate $B_t^H$ under the usual quadratic risk is

$$E_\theta \left[ \int_0^T \left( \|g(B_t^H, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g_i(B_t^H, t) \right) dt \right] < 0.$$  

As an application, take $g$ of the form

$$g(x, t) = at^{2H} \frac{\|x\|^2}{\|x\|^2}x,$$  \hspace{1cm} (4.12)

where $a$ is a constant strictly positive and $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is bounded derivable function. The next lemma give a sufficient condition for $g$ in (4.12) to satisfies the assumption (A).

**Lemma 1** If $d \geq 3$ and $H < \frac{1}{2}$ then

$$E \left[ \int_0^T \frac{1}{\|B_t^H\|^2} dt \right] < \infty.$$  \hspace{1cm} (4.13)

**Proof:** Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)((t, w); B_t^H(w) = 0) = 0$$

where $(dt \times P)$ is the product measure. Using the change of variable and $d \geq 3$ we see that

$$E \int_0^T \frac{1}{\|B_t^H\|^2} dt = \int_0^T \frac{dt}{t^{2H}} \int_{\mathbb{R}^d} \frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi} \|y\|^2} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,$$

where $C$ is a constant depending only on $d$. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

**Theorem 3** Assume that $d \geq 3$. If the function $r$, the constant $a$ and the parameter $H$ satisfy:

i) $0 \leq r(\cdot) \leq 1$

ii) $r(\cdot)$ is differentiable and increasing
iii) $0 < a \leq 2(d - 2)$ and $H < 1/2$, then the estimator

$$\delta(B^H_t) = B^H_t - at^{2H} \frac{r(||B^H_t||^2)}{||B^H_t||^2} B^H_t, \quad t \in [0, T].$$

dominates $B^H_t$.

Proof: It suffices to prove that $\Delta R(\theta) < 0$. From (4.11) and the hypothesis i) and ii) we can write

$$\Delta R(\theta) = aE_\theta \left[ \int_0^T t^{4H} \left( \frac{ar^2(||B^H_t||^2)}{||B^H_t||^2} - 2(d - 2) \frac{r(||B^H_t||^2)}{||B^H_t||^2} \right) dt \right]$$

$$\leq a[a - 2(d - 2)] E_\theta \left[ a \int_0^T t^{4H} \frac{r(||B^H_t||^2)}{||B^H_t||^2} \right].$$

Combining this fact with the assumption iii) yields that the risk difference is negative. Which proves the desired result.

For $r = 1$, we obtain a James-Stein type estimator:

**Corollary 2** Let $d \geq 3$, $0 < H < \frac{1}{2}$ and $0 < a \leq 2(d - 2)$. Then the estimator

$$\left(1 - \frac{at^{2H}}{||B^H_t||^2}\right) B^H_t, \quad t \in [0, T]$$

dominates $B^H_t$.

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