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Estimation of the drift of fractional Brownian motion

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Abstract

We consider the problem of efficient estimation for the drift of fractional Brownian motion $B^H := (B^H_t)_{t \in [0,T]}$ with hurst parameter $H$ less than $\frac{1}{2}$. We also construct super-efficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

Key words : Fractional Brownian Motion, Stein estimate, MLE

2000 Mathematics Subject Classification: 60G15, 62G05, 62B05, 62M09.

1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \{(B^H_{t,i} ; t \in [0,T])\}$ be a $d$-dimensional fractional Brownian motion (fBm) defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, $B^H$ is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for every $i = 1,\ldots,d$ $B^H_{t,i}$ is a Gaussian process and covariance function given by

$$E(B^H_{s,i}B^H_{t,i}) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s,t \in [0,T].$$

For each $i = 1,\ldots,d$, $(\mathcal{F}_t)_{t \in [0,T]}$ denotes the filtration generated by $(B^H_{t,i})_{t \in [0,T]}$.

The fBm was first introduced by [3] and studied by [4]. Notice that if $H = \frac{1}{2}$, the
process $B^H$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let $M$ be a subspace of the Cameron-Martin space defined by

$$M = \left\{ \varphi : [0, T] \to \mathbb{R}^d; \varphi_t = \int_0^t \dot{\varphi}_s ds \text{ with } \varphi_t \in L^2([0, T]) \text{ and } \varphi_t \in \mathcal{I}^{H+i\frac{1}{2}}(L^2([0, T])), i = 1, \ldots, d \right\}.$$  

Let $\theta = \left\{ (\theta^1_t, \ldots, \theta^d_t); t \in [0, T] \right\}$ be a function belonging to $M$. Then, applying Girsanov theorem (see Theorem 2 in [9]), there exists a probability measure absolutely continuous with respect to $P$ under which the process $\tilde{B}^H$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0, T]$$  

is a fBm with Hurst parameter $H$ and mean zero. In this case, we say that, under the probability $P_\theta$, the process $B^H$ is a fBm with drift $\theta$.

We consider in this paper the problem of estimating the drift $\theta$ of $B^H$ under the probability $P_\theta$, with Hurst parameter $H < 1/2$. We wish to estimate $\theta$ under the usual quadratic risk, that is defined for any estimator $\delta$ of $\theta$ by

$$\mathcal{R}(\theta, \delta) = E_\theta \left[ \int_0^T |\delta_t - \theta_t|^2 dt \right]$$

where $E_\theta$ is the expectation with respect to a probability $P_\theta$.

Let $X = (X^1, \ldots, X^d)$ be a normal vector with mean $\theta = (\theta^1, \ldots, \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of $\theta$ is $X$ itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by $X$. That is

$$\sigma^2 d = E \left[ \|X - \theta\|_d^2 \right] = \inf_{\xi \in S} E \left[ \|\xi - \theta\|_d^2 \right],$$

where $S$ is the class of unbiased estimators of $\theta$ and $\| \cdot \|_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

[12] constructed biased superefficient estimators of $\theta$ of the form

$$\delta_{a,b}(X) = \left( 1 - \frac{b}{a + \|X\|^2} \right) X$$

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for $a$ sufficiently small and $b$ sufficiently large when $d \geq 3$. 

Recently, an infinite-dimensional extension of this result has been given by 

\[ \hat{\theta} = (X_t)_{t \in [0,T]} \] of a continuous Gaussian martingale $(X_t)_{t \in [0,T]}$ with quadratic variation $\sigma_t^2 dt$, where $\sigma \in L^2([0,T], dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t \in [0,T]}$ is an efficient estimator of $(\theta_t)_{t \in [0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian process of the form:

\[ X_t := \int_0^t K(t,s) dW_s, \quad t \in [0,T], \]

where $(W_t)_{t \in [0,T]}$ is a standard Brownian motion and $K(\ldots)$ is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where $F$ is a positive superharmonic random variable and $D$ is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\hat{\theta} = B^H$ is an efficient estimator of $\theta$ under the probability $P_\theta$ with risk

\[ \mathcal{R}(\theta, B^H) = E_\theta \left[ \int_0^T \| B^H_t - \theta_t \|^2 dt \right] = \frac{T^{2H+1}}{2H+1} d. \]

Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of $\theta$.

In Section 4, we construct a class of biased estimators of James-Stein type of the form

\[ \delta(B^H)_t = \left( 1 - at^{2H} \left( \frac{r(\|B^H\|)}{\|B^H\|^2} \right) \right) B^H_t, \quad t \in [0,T]. \]

We give sufficient conditions on the function $r$ and on the constant $a$ in order that $\delta(B^H)$ dominates $B^H$ under the usual quadratic risk i.e.

\[ \mathcal{R}(\theta, \delta(B^H)) < \mathcal{R}(\theta, B^H) \quad \text{for all } \theta \in M. \quad (1.2) \]

2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.
The fractional Brownian motion $B^H$ has the following stochastic integral representation (see for instance, [1], [8])

\[ B^H_t = \int_0^t K_H(t, s)dW^i_s, \quad i = 1, \ldots, d; \quad t \in [0, T] \quad (2.3) \]

where $W = (W^1, \ldots, W^d)$ denotes the $d$-dimensional Brownian motion and the kernel $K_H(t, s)$ is equal to

\[
    c_H(t - s)^{H - \frac{1}{2}} + c_H \left( \frac{1}{2} - H \right) \int_s^t (u - s)^{H - \frac{3}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2} - H} \right) du \quad \text{if } H \leq \frac{1}{2}, \\
    c_H \left( H - \frac{1}{2} \right) \int_s^t (u - s)^{H - \frac{3}{2}} \left( \frac{s}{u} \right)^{H - \frac{1}{2}} du \quad \text{if } H > \frac{1}{2},
\]

if $s < t$ and $K_H(t, s) = 0$ if $s \geq t$. Here $c_H$ is the normalizing constant

\[
    c_H = \sqrt{\frac{2H \Gamma\left( \frac{3}{2} - H \right)}{\Gamma\left( H + \frac{1}{2} \right) \Gamma(2 - 2H)}}
\]

where $\Gamma$ is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [3].

The left fractional Riemann-Liouville integral of $f \in L^1((a, b))$ of order $\alpha > 0$ on $(a, b)$ is given at almost all $x \in (a, b)$ by

\[
    I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y)dy.
\]

If $f \in L^p((a, b))$ with $0 < \alpha < 1$ and $p > 1$ then the left-sided Riemann-Liouville derivative of $f$ of order $\alpha$ defined by

\[
    D^\alpha_{a^+} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x - a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x - y)^{\alpha+1}} dy \right)
\]

for almost all $x \in (a, b)$.

For $H \in (0, 1)$, the integral transform

\[
    (K_H f)(t) = \int_0^t K_H(t, s)f(s)ds
\]
is a isomorphism from $L^2([0,1])$ onto $L^2([0,1])$ and its inverse operator $K_H^{-1}$ is given by

$$K_H^{-1}f = t^{H-rac{1}{2}}D_{0+}^H t^{\frac{1}{2}-H} f' \quad \text{for } H > 1/2, \quad (2.4)$$

$$K_H^{-1}f = t^{\frac{1}{2}-H}D_{0+}^{2H} t^{H-rac{1}{2}}D_{0+}^H f \quad \text{for } H < 1/2. \quad (2.5)$$

Moreover, for $H < \frac{1}{2}$, if $f$ is an absolutely continuous function then $K_H^{-1}f$ can be represented of the form (see [4])

$$K_H^{-1}f = t^{H-rac{1}{2}}D_{0+}^H t^{\frac{1}{2}-H} f'. \quad (2.6)$$

3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function $\theta = (\theta^1, \ldots, \theta^d)$ belonging to $M$. An estimator $\xi = (\xi^1, \ldots, \xi^d)$ of $\theta = (\theta^1, \ldots, \theta^d)$ is called unbiased if, for every $t \in [0,T]$

$$E_\theta(\xi^i_t) = \theta^i_t, \quad i = 1, \ldots, d$$

and it is called adapted if, for each $i = 1, \ldots, d$, $\xi^i$ is adapted to $(\mathcal{F}_t^i)_{t \in [0,T]}$.

Since for any $i = 1, \ldots, d$, the function $\theta^i$ is deterministic

$$\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem yields that there exists a probability measure $P_\theta$ absolutely continuous with respect to $P$ under which the process $\tilde{B}^H := (\tilde{B}^H_t; t \in [0,T])$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0,T] \quad (3.7)$$

is a d-dimensional fBm with Hurst parameter $H$ and mean zero. Moreover the Girsanov density of $P_\theta$ with respect to $P$ is given by:

$$\frac{dP_\theta}{dP} = \exp \left[ \sum_{i=1}^d \left( \int_0^T K_H^{-1}(\theta^i)(s)dW^i_s - \frac{1}{2} \int_0^T (K_H^{-1}(\theta^i)(s))^2 ds \right) \right]$$
and
\[ \tilde{B}_t^H = \int_0^t K_H(t, s)d\tilde{W}_s \]
where \( \tilde{W} \) is a d-dimensional Brownian motion under the probability \( P_\theta \) and
\[ \tilde{W}_t^i = W_t^i - \int_0^t K_{H^{-1}(\theta^i)}(s)ds, \quad i = 1, \ldots, d; \quad t \in [0, T]. \]
The equation (3.7) implies that \( B^H \) is an unbiased and adapted estimator of \( \theta \) under probability \( P_\theta \). In addition, we obtain the Cramer-Rao type bound:
\[ R(H, \hat{\theta}) := R(\theta, B^H) = \int_0^T E_{\theta} \| \tilde{B}_t^H \|^2 dt = d \int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d. \]

The first main result of this section is given by the following proposition which asserts that \( \hat{\theta} = B^H \) is an efficient estimator of \( \theta \).

**Theorem 1** Assume that \( H < \frac{1}{2} \). If \( \xi \) is an unbiased and adapted estimator of \( \theta \), then
\[ E_{\theta} \int_0^T \| \xi_t - \theta_t \|^2 dt \geq R(H, \hat{\theta}). \]  
(3.8)

**Proof:** Since \( \xi \) is unbiased, then for every \( \varphi \in M \) we have
\[ E_{\varphi}(\xi_t^j) = E_{\varphi}(\varphi_t^j), \quad j = 1, \ldots, d. \]
Let \( \varphi = \theta + \varepsilon \psi \) with \( \psi \in M \) and \( \varepsilon \in \mathbb{R} \). Then for every \( t \in [0, T] \) and \( j \in \{1, \ldots, d\} \), we have
\[ E_{\theta + \varepsilon \psi}(\xi_t^j) = E_{\theta + \varepsilon \psi}(\theta_t^j + \varepsilon \psi_t^j) = E_{\theta + \varepsilon \psi}(\theta_t^j) + \varepsilon \psi_t^j. \]
This implies that for every $j = 1, \ldots, d$

$$
\psi^j_t = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} E_{\theta+\varepsilon\psi}(\xi^j_t - \theta^j_t) \\
= E \left( \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \exp \left[ \sum_{i=1}^d \left( \int_0^t K^{-1}_H(\theta^i + \varepsilon\psi^i)(s)dW^i_s \right. \right. \\
\left. \left. - \frac{1}{2} \int_0^t (K^{-1}_H(\theta^i + \varepsilon\psi^i)(s))^2 ds \right) \right] \right) (\xi^j_t - \theta^j_t) \\
= E \left( \sum_{i=1}^d \left[ \int_0^t K^{-1}_H(\psi^i)(s)dW^i_s - \int_0^t K^{-1}_H(\psi^i)(s)K^{-1}_H(\theta^i)(s)ds \right] \right) \\
\times (\xi^j_t - \theta^j_t) \\
= E \left( \sum_{i=1}^d \left[ \int_0^t K^{-1}_H(\psi^i)(s)d\tilde{W}^i_s \right] (\xi^j_t - \theta^j_t) \right) \\
= E \left( \left[ \int_0^t K^{-1}_H(\psi^j)(s)d\tilde{W}^j_s \right] (\xi^j_t - \theta^j_t) \right). 
$$

Applying Cauchy-Schwarz inequality in $L^2(\Omega, dP_\theta)$, we obtain that for every $t \in [0, T]$

$$
||\psi_t||^2 = \sum_{j=1}^d (\psi^j_t)^2 \leq \sum_{j=1}^d E_{\theta} \left( (\xi^j_t - \theta^j_t)^2 \right) E_{\theta} \left( \left[ \int_0^t K^{-1}_H(\psi^j)(s)d\tilde{W}^j_s \right]^2 \right) \\
= \sum_{j=1}^d E_{\theta} \left[ (\xi^j_t - \theta^j_t)^2 \int_0^t (K^{-1}_H(\psi^j)(s))^2 ds \right].
$$

We take for each $j = 1, \ldots, d$, $\psi^j_t = t^{2H}$. Since $t \rightarrow t^{2H}$ is absolutely continuous function, then by (2.6), a simple calculation shows that

$$
K^{-1}_H(t^{2H}) = 2^{H}t^{H-\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2} - H)} t^{H-\frac{1}{2}} \\
= 2^{H} \frac{\beta(\frac{1}{2} - H, H + \frac{1}{2})}{\Gamma(\frac{1}{2} - H)} t^{H-1/2} \\
= 2^{H} (\Gamma(\frac{1}{2} + H)) t^{H-1/2}.
$$

It is known that

$$
0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1, 2]. \quad (3.9)
$$
Combining the facts that \( z \Gamma(z) = \Gamma(z + 1), \) \( z > 0, \) \( 2H \leq (H + \frac{1}{2})^2 \) and (3.9), we obtain
\[
\text{d}t^{2H} = \|\psi_t\|^2 \leq (\Gamma(\frac{3}{2} + H))^2 E_\theta (\|\xi_t - \theta_t\|^2) \leq E_\theta (\|\xi_t - \theta_t\|^2).
\]
Hence, by an integration with respect to \( \text{d}t, \) we get
\[
R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 \text{d}t.
\]
Therefore (3.8) is satisfied.

**Corollary 1** The process \( \hat{\theta} = B^H \) is a maximum likelihood estimator of \( \theta. \)

**Proof:** We have for every \( \psi \in M \)
\[
\frac{d}{d\varepsilon} \varepsilon = 0 \exp \left[ \sum_{i=1}^{d} \int_0^t K_H^{-1}(\hat{\theta}^i + \varepsilon \psi^i)(s) dW_s^i - \frac{1}{2} \int_0^t (K_H^{-1}(\hat{\theta}^i + \varepsilon \psi^i)(s))^2 ds \right] = 0.
\]
Hence
\[
\sum_{i=1}^{d} \left( \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s) ds \right) = 0.
\]
Which implies that for every \( i = 1, \ldots, d \)
\[
E \left( \int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s) ds \right)^2 = 0.
\]
Then, for each \( i = 1, \ldots, d \)
\[
W_t^i = \int_0^t K_H^{-1}(\hat{\theta}^i)(s) ds, \quad t \in [0, T].
\]
Therefore by (2.3), we obtain that \( B^H = \hat{\theta}. \)

## 4 Superefficient James-Stein type estimators

The aim of this section is to construct superefficient estimators of \( \theta \) which dominate, under the usual quadratic risk, the natural MLE estimator \( B^H. \) The class of estimators considered here are of the form
\[
\delta(B^H)_t = B^H_t + g(B^H_t, t), \quad t \in [0, T]
\]
(4.10)
where $g : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is a function. The problem turns to find sufficient conditions on $g$ such that $\mathcal{R}(\theta, \delta(B^H)) < \infty$ and the risk difference is negative, i.e.

$$\Delta \mathcal{R}(\theta) = \mathcal{R}(\theta, \delta(B^H)) - \mathcal{R}(\theta, B^H) < 0.$$ 

In the sequel we assume that the function $g$ satisfies the following assumption:

\begin{align*}
(A) \quad & \left\{ \begin{array}{l}
E_\theta \left[ \int_0^T ||g(B^H_t, t)||_d^2 \, dt \right] < \infty, \\
& \text{the partial derivatives } \partial_ig^i := \frac{\partial g^i}{\partial x^i}, \ i = 1, \ldots, n \text{ of } g \text{ exist.}
\end{array} \right.
\end{align*}

Then $\mathcal{R}(\theta, \delta(B^H)) < \infty$. Moreover

$$\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left| \right| |B^H_t + g(B^H_t, t) - \theta_t|_d^2 - |B^H_t - \theta_t|_d^2 \right] dt \right] = E_\theta \left[ \int_0^T \left| \right| g(B^H_t, t)|_d^2 + 2 \sum_{i=1}^d \left( g^i(B^H_t, t)(B^H_{t,i} - \theta^i_t) \right) dt \right].$$

In addition,

\begin{align*}
E_\theta & \int_0^T \sum_{i=1}^d \left( g^i(B^H_t, t)(B^H_{t,i} - \theta^i_t) \right) dt \\
= & \sum_{i=1}^d \int_0^T (2\pi t^2H)^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} g^i(x^1, \ldots, x^d, t)(x^i - \theta^i_t) \right.
& \left. \times e^{-\frac{\sum_{j=1}^d (x^j - \theta^j_t)^2}{2t^2H}} dx^1 \ldots dx^d \right) dt \\
= & \sum_{i=1}^d \int_0^T (2\pi t^2H)^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t) \right.
& \left. \times e^{-\frac{\sum_{j=1}^d (x^j - \theta^j_t)^2}{2t^2H}} dx^1 \ldots dx^d \right) dt \\
= & \sum_{i=1}^d \int_0^T (t^{2H} E_\theta \partial_i g^i(B^H_t, t)) dt = E_\theta \left[ \sum_{i=1}^d \int_0^T t^{2H} \partial_i g^i(B^H_t, t) dt \right].
\end{align*}

Consequently, the risk difference equals

$$\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( \left| \right| g(B^H_t, t) \right|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B^H_t, t) \right) dt \right]. \quad (4.11)$$
We can now state the following theorem.

**Theorem 2** Let \( g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) be a function satisfying (A). A sufficient conditions for the estimator \( (B^H_t + g(B^H_t, t))_{t \in [0,T]} \) to dominate \( B^H_t \) under the usual quadratic risk is

\[
E_{\theta} \left[ \int_0^T \left( \|g(B^H_t, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B^H_t, t) \right) dt \right] < 0.
\]

As an application, take \( g \) of the form

\[
g(x, t) = at^{2H} \frac{\|x\|^2}{\|x\|^2} - x, \tag{4.12}
\]

where \( a \) is a constant strictly positive and \( r : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is bounded derivable function.

The next lemma give a sufficient condition for \( g \) in (4.12) to satisfies the assumption (A).

**Lemma 1** If \( d \geq 3 \) and \( H < \frac{1}{2} \) then

\[
E \left[ \int_0^T \frac{1}{\|B^H_t\|^2} dt \right] < \infty. \tag{4.13}
\]

**Proof:** Firstly the integral given by (4.13) is well defined, because

\[
(dt \times P)((t, w); B^H_t(w) = 0) = 0
\]

where \((dt \times P)\) is the product measure.

Using the change of variable and \( d \geq 3 \) we see that

\[
E \int_0^T \frac{1}{\|B^H_t\|^2} dt = \int_0^T dt \int_{\mathbb{R}^d} \frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi\|y\|^2}} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,
\]

where \( C \) is a constant depending only on \( d \). Furthermore, since \( H < \frac{1}{2} \) then (4.13) holds.

**Theorem 3** Assume that \( d \geq 3 \). If the function \( r \), the constant \( a \) and the parameter \( H \) satisfy:

i) \( 0 \leq r(\cdot) \leq 1 \)

ii) \( r(\cdot) \) is differentiable and increasing
iii) $0 < a \leq 2(d-2)$ and $H < 1/2$,

then the estimator

$$\delta(B^H_t) = B^H_t - at^{2H} \frac{r(||B^H_t||^2)}{||B^H_t||^2} B^H_t, \quad t \in [0, T].$$

dominates $B^H$.

**Proof:** It suffices to prove that $\Delta \mathcal{R}(\theta) < 0$. From (4.11) and the hypothesis i) and ii) we can write

$$\Delta \mathcal{R}(\theta) = aE_\theta \left[ \int_0^T t^{4H} \left( \frac{ar^2(||B^H_t||^2)}{||B^H_t||^2} - 2(d-2)\frac{r(||B^H_t||^2)}{||B^H_t||^2} ight) - 4r'(||B^H_t||^2) \right] dt$$

$$\leq a[a - 2(d-2)] E_\theta \left[ a \int_0^T t^{4H} \frac{r(||B^H_t||^2)}{||B^H_t||^2} \right].$$

Combining this fact with the assumption iii) yields that the risk difference is negative. Which proves the desired result.

For $r = 1$, we obtain a James-Stein type estimator:

**Corollary 2** Let $d \geq 3$, $0 < H < \frac{1}{2}$ and $0 < a \leq 2(d-2)$. Then the estimator

$$\left(1 - \frac{at^{2H}}{||B^H_t||^2}\right) B^H_t, \quad t \in [0, T]$$

dominates $B^H$.

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