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Submitted on 8 May 2009

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Estimation of the drift of fractional Brownian motion

Khalifa Es-Sebaiy 1,3, Idir Ouassou 2, Youssef Ouknine 1

1 Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University 2390 Marrakesh, Morocco.
2 ENSA, Cadi Ayyad University, Marrakesh, Morocco,
3 SAMOS/MATISSE, Centre d’Economie de La Sorbonne, Université de Panthéon-Sorbonne Paris 1, 90, rue de Tolbiac, 75634 Paris Cedex 13, France.

May 8, 2009

Abstract

We consider the problem of efficient estimation for the drift of fractional Brownian motion $B^H := \{B^H_t\}_{t \in [0,T]}$ with hurst parameter $H$ less than $\frac{1}{2}$. We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

Key words : Fractional Brownian Motion, Stein estimate, MLE

2000 Mathematics Subject Classification: 60G15, 62G05, 62B05, 62M09.

1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \{(B^H_t, \ldots, B^H_d); t \in [0,T]\}$ be a $d$-dimensional fractional Brownian motion (fBm) defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, $B^H$ is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for every $i = 1, \ldots, d$ $B^{H,i}$ is a Gaussian process and covariance function given by

$$E(B^H_s; B^H_t) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, T].$$

For each $i = 1, \ldots, d$, $(\mathcal{F}_t)_{t \in [0,T]}$ denotes the filtration generated by $(B^H_t)_{t \in [0,T]}$.

The fBm was first introduced by [3] and studied by [4]. Notice that if $H = \frac{1}{2}$, the
process $B^H$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let $M$ be a subspace of the Cameron-Martin space defined by

$$M = \left\{ \varphi : [0,T] \to \mathbb{R}^d; \varphi_t = \int_0^t \dot{\varphi}_s ds \text{ with } \dot{\varphi} \in L^2([0,T]) \right\}$$

and $\varphi^i \in \mathcal{H}^{1/2} + \left(L^2([0,T])\right)_i, i = 1, ..., d$.

Let $\theta = \{(\theta^i_1, ..., \theta^i_d); t \in [0,T]\}$ be a function belonging to $M$. Then, Applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to $P$ under which the process $\tilde{B}^H$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0,T]$$

(1.1)

is a fBm with Hurst parameter $H$ and mean zero. In this case, we say that, under the probability $P_{\theta}$, the process $B^H$ is a fBm with drift $\theta$.

We consider in this paper the problem of estimating the drift $\theta$ of $B^H$ under the probability $P_{\theta}$, with Hurst parameter $H < 1/2$. We wish to estimate $\theta$ under the usual quadratic risk, that is defined for any estimator $\delta$ of $\theta$ by

$$\mathcal{R}(\theta, \delta) = E_{\theta} \left[ \int_0^T |\delta_t - \theta_t|^2 dt \right]$$

where $E_{\theta}$ is the expectation with respect to a probability $P_{\theta}$.

Let $X = (X^1, ..., X^d)$ be a normal vector with mean $\theta = (\theta^1, ..., \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of $\theta$ is $X$ itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by $X$. That is

$$\sigma^2 d = E \left[ \|X - \theta\|_d^2 \right] = \inf_{\xi \in \mathcal{S}} E \left[ \|\xi - \theta\|_d^2 \right],$$

where $\mathcal{S}$ is the class of unbiased estimators of $\theta$ and $\|\cdot\|_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

[12] constructed biased superefficient estimators of $\theta$ of the form

$$\delta_{a,b}(X) = \left( 1 - \frac{b}{a + \|X\|^2} \right) X$$
for $a$ sufficiently small and $b$ sufficiently large when $d \geq 3$. They sharpened later this result and presented an explicit class of biased superefficient estimators of the form
\[
\left(1 - \frac{a}{\|X\|^2_d}\right)X, \text{ for } 0 < a < 2(d - 2).
\]

Recently, an infinite-dimensional extension of this result has been given by [10]. The authors constructed unbiased estimators of the drift $(\theta_t)_{t \in [0,T]}$ of a continuous Gaussian martingale $(X_t)_{t \in [0,T]}$ with quadratic variation $\sigma^2_t dt$, where $\sigma \in L^2([0,T],dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t \in [0,T]}$ is an efficient estimator of $(\theta_t)_{t \in [0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian process of the form:
\[
X_t := \int_0^t K(t,s) dW_s, \quad t \in [0,T],
\]
where $(W_t)_{t \in [0,T]}$ is a standard Brownian motion and $K(\cdot,\cdot)$ is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where $F$ is a positive superharmonic random variable and $D$ is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\hat{\theta} = B^H$ is an efficient estimator of $\theta$ under the probability $P_\theta$ with risk
\[
\mathcal{R}(\theta, B^H) = E_\theta \left[ \int_0^T \|B^H_t - \theta_t\|^2 dt \right] = \frac{T^{2H+1}}{2H+1}. d.
\]
Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of $\theta$.

In Section 4, we construct a class of biased estimators of James-Stein type of the form
\[
\delta(B^H)_t = \left(1 - at^{2H} \left( \frac{r(\|B^H\|^2)}{\|B^H\|^2} \right) \right) B^H_t, \quad t \in [0,T].
\]
We give sufficient conditions on the function $r$ and on the constant $a$ in order that $\delta(B^H)$ dominates $B^H$ under the usual quadratic risk i.e.
\[
\mathcal{R}(\theta, \delta(B^H)) < \mathcal{R}(\theta, B^H) \quad \text{ for all } \theta \in M. \quad (1.2)
\]

2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.
The fractional Brownian motion $B^H$ has the following stochastic integral representation (see for instance, [1], [8])

$$B^H_i = \int_0^t K_H(t, s) dW^i_s, \quad i = 1, \ldots, d; \quad t \in [0, T]$$  \hspace{1cm} (2.3)

where $W = (W^1, \ldots, W^d)$ denotes the $d$-dimensional Brownian motion and the kernel $K_H(t, s)$ is equal to

$$c_H(t-s)^{H-\frac{1}{2}} + c_H \left(\frac{1}{2} - H\right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du \quad \text{if} \quad H \leq \frac{1}{2},$$

$$c_H (H-\frac{1}{2}) \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{\frac{1}{2}-H} du \quad \text{if} \quad H > \frac{1}{2},$$

if $s < t$ and $K_H(t, s) = 0$ if $s \geq t$. Here $c_H$ is the normalizing constant

$$c_H = \sqrt{\frac{2H^{\frac{3}{2}} - H}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}}$$

where $\Gamma$ is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [3].

The left fractional Riemann-Liouville integral of $f \in L^1((a, b))$ of order $\alpha > 0$ on $(a, b)$ is given at almost all $x \in (a, b)$ by

$$I^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy.$$  \hspace{1cm} (2.4)

If $f \in L^p_a((a, b))$ with $0 < \alpha < 1$ and $p > 1$ then the left-sided Riemann-Liouville derivative of $f$ of order $\alpha$ defined by

$$D^\alpha_a f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy\right)$$

for almost all $x \in (a, b)$.

For $H \in (0, 1)$, the integral transform

$$(K_H f)(t) = \int_0^t K_H(t, s) f(s) ds$$
is an isomorphism from $L^2([0,1])$ onto $L^2_{0+}([0,1])$ and its inverse operator $K^{-1}_H$ is given by

$$K^{-1}_H f = t^{H-1} D^{1-H}_{0+} f'$$ for $H > 1/2$, \hspace{1cm} (2.4)
$$K^{-1}_H f = t^{1-H} D^H_{0+} f$$ for $H < 1/2$, \hspace{1cm} (2.5)

Moreover, for $H < 1/2$, if $f$ is an absolutely continuous function then $K^{-1}_H f$ can be represented of the form (see [10])

$$K^{-1}_H f = t^{H-1} D^{1-H}_{0+} f'$$.

3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function $\theta = (\theta^1, \ldots, \theta^d)$ belonging to $M$. An estimator $\xi = (\xi^1, \ldots, \xi^d)$ of $\theta = (\theta^1, \ldots, \theta^d)$ is called unbiased if, for every $t \in [0,T]$

$$E_{\theta}(\xi^i_t) = \theta^i_t, \hspace{1cm} i = 1, \ldots, d$$

and it is called adapted if, for each $i = 1, \ldots, d$, $\xi^i$ is adapted to $(\mathcal{F}_t^i)_{t \in [0,T]}$.

Since for any $i = 1, \ldots, d$, the function $\theta^i$ is deterministic and

$$\int_0^T (K^{-1}_H (\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem yields that there exists a probability measure $P_\theta$ absolutely continuous with respect to $P$ under which the process $\tilde{B}^H := (\tilde{B}^H_t; t \in [0,T])$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \hspace{1cm} t \in [0,T]$$

is a d-dimensional fBm with Hurst parameter $H$ and mean zero. Moreover the Girsanov density of $P_\theta$ with respect to $P$ is given by:

$$\frac{dP_\theta}{dP} = \exp \left[ \sum_{i=1}^d \left( \int_0^T K^{-1}_H (\theta^i)(s) dW^i_s - \frac{1}{2} \int_0^T (K^{-1}_H (\theta^i)(s))^2 ds \right) \right]$$
and
\[ \tilde{B}_t^H = \int_0^t K_H(t, s)d\tilde{W}_s \]

where \( \tilde{W} \) is a \( d \)-dimensional Brownian motion under the probability \( P_\theta \) and
\[ \tilde{W}_t^i = W_t^i - \int_0^t K_H^{-1}(\theta^i)(s)ds, \quad i = 1, \ldots, d; \quad t \in [0, T]. \]

The equation (3.7) implies that \( B^H \) is an unbiased and adapted estimator of \( \theta \) under probability \( P_\theta \). In addition, we obtain the Cramer-Rao type bound:
\[ R(H, \hat{\theta}) := R(\theta, B^H) = \int_0^T E_\theta \| \tilde{B}_t^H \|^2 dt = \frac{T^{2H+1}}{2H+1}d. \]

The first main result of this section is given by the following proposition which asserts that \( \hat{\theta} = B^H \) is an efficient estimator of \( \theta \).

**Theorem 1** Assume that \( H < \frac{1}{2} \). If \( \xi \) is an unbiased and adapted estimator of \( \theta \), then
\[ E_\theta \int_0^T \| \xi_t - \theta_t \|^2 dt \geq R(H, \hat{\theta}). \]  

**Proof:** Since \( \xi \) is unbiased, then for every \( \varphi \in M \) we have
\[ E_\varphi(\xi_t^j) = E_\varphi(\varphi_t^j), \quad j = 1, \ldots, d. \]

Let \( \varphi = \theta + \varepsilon \psi \) with \( \psi \in M \) and \( \varepsilon \in \mathbb{R} \). Then for every \( t \in [0, T] \) and \( j \in \{1, \ldots, d\} \), we have
\[ E_{\theta+\varepsilon\psi}(\xi_t^j) = E_{\theta+\varepsilon\psi}(\theta_t^j + \varepsilon\psi_t^j) = E_{\theta+\varepsilon\psi}(\theta_t^j) + \varepsilon\psi_t^j. \]
This implies that for every \( j = 1, \ldots, d \)

\[
\psi_t^j = \frac{d}{d\varepsilon_{\varepsilon=0}} E_{\theta + \varepsilon\psi}(\xi_t^j - \theta_t^j)
\]

\[
= E \left( \frac{d}{d\varepsilon_{\varepsilon=0}} \exp \left( \sum_{i=1}^d \left( \int_0^t K_H^{-1}(\theta^i + \varepsilon\psi^i(s))dW_s^i \right) \right) \right) \left( \xi_t^j - \theta_t^j \right)
\]

\[
= E \left( \sum_{i=1}^d \left( \int_0^t K_H^{-1}(\psi^i(s))dW_s^i \right) \left( \xi_t^j - \theta_t^j \right) \right)
\]

\[
= E \left( \left( \int_0^t K_H^{-1}(\psi^j(s))d\tilde{W}_s^j \right) \left( \xi_t^j - \theta_t^j \right) \right)
\]

Applying Cauchy-Schwarz inequality in \( L^2(\Omega, dP_\theta) \), we obtain that for every \( t \in [0, T] \)

\[
||\psi_t||^2 = \sum_{j=1}^d (\psi_t^j)^2 \leq \sum_{j=1}^d E_\theta \left( (\xi_t^j - \theta_t^j)^2 \right) E_\theta \left( \left( \int_0^t K_H^{-1}(\psi^j(s))s d\tilde{W}_s^j \right)^2 \right)
\]

\[
= \sum_{j=1}^d E_\theta \left[ (\xi_t^j - \theta_t^j)^2 \int_0^t (K_H^{-1}(\psi^j(s))^2 ds \right].
\]

We take for each \( j = 1, \ldots, d \), \( \psi_t^j = t^{2H} \). Since \( t \rightarrow t^{2H} \) is absolutely continuous function, then by (2.6), a simple calculation shows that

\[
K_H^{-1}(t^{2H}) = 2Ht^{H-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + H)}{\Gamma(\frac{1}{2})} t^{-\frac{1}{2} - H}
\]

\[
= 2H \beta(\frac{1}{2} - H, H + \frac{1}{2}) t^{H-1/2}
\]

\[
= 2H(\Gamma(\frac{1}{2} + H)) t^{H-1/2}.
\]

It is known that

\[
0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1, 2]. \tag{3.9}
\]
Combining the facts that \( z \Gamma(z) = \Gamma(z + 1), \ z > 0, \ 2H \leq (H + \frac{1}{2})^2 \) and (3.9), we obtain
\[
dt^{2H} = \|\psi_t\|^2 \leq (\Gamma(\frac{3}{2} + H))^2 E_\theta(\|\xi_t - \theta_t\|^2) \leq E_\theta(\|\xi_t - \theta_t\|^2).
\]
Hence, by an integration with respect to \( dt \), we get
\[
R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H + 1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt.
\]
Therefore (3.8) is satisfied.

**Corollary 1** The process \( \hat{\theta} = B^H \) is a maximum likelihood estimator of \( \theta \).

**Proof:** We have for every \( \psi \in M \)
\[
\frac{d}{d\varepsilon}_{\varepsilon=0} \exp \left[ \sum_{i=1}^d \int_0^t K_H^{-1}(\hat{\psi}^i + \varepsilon \psi^i)(s)dW_s^i - \frac{1}{2} \int_0^t (K_H^{-1}(\hat{\psi}^i + \varepsilon \psi^i)(s))^2 ds \right] = 0.
\]
Hence
\[
\sum_{i=1}^d \left( \int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\psi}^i)(s)ds \right) = 0.
\]
Which implies that for every \( i = 1, \ldots, d \)
\[
E \left( \int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\psi}^i)(s)ds \right)^2 = 0.
\]
Then, for each \( i = 1, \ldots, d \)
\[
W_t^i = \int_0^t K_H^{-1}(\hat{\psi}^i)(s)ds, \quad t \in [0, T].
\]
Therefore by (2.3), we obtain that \( B^H = \hat{\theta} \).

**4 Superefficient James-Stein type estimators**

The aim of this section is to construct superefficient estimators of \( \theta \) which dominate, under the usual quadratic risk, the natural MLE estimator \( B^H \). The class of estimators considered here are of the form
\[
\delta(B^H)_t = B^H_t + g(B^H_t, t), \quad t \in [0, T]
\]
(4.10)
where \( g: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) is a function. The problem turns to find sufficient conditions on \( g \) such that \( \mathcal{R}(\theta, \delta(B^H)) < \infty \) and the risk difference is negative, i.e.

\[
\Delta \mathcal{R}(\theta) = \mathcal{R}(\theta, \delta(B^H)) - \mathcal{R}(\theta, B^H) < 0.
\]

In the sequel we assume that the function \( g \) satisfies the following assumption:

\[
(A) \quad \left\{ \begin{array}{l}
E_\theta \left[ \int_0^T \| g(B^H_t, t) \|_{\mathbb{R}^d}^2 dt \right] < \infty, \\
\text{the partial derivatives } \partial_i g_i := \frac{\partial g_i}{\partial x^i}, \ i = 1, \ldots, n \text{ of } g \text{ exist}.
\end{array} \right.
\]

Then \( \mathcal{R}(\theta, \delta(B^H)) < \infty \). Moreover

\[
\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( \| B^H_t - \theta_i \|_{\mathbb{R}^d}^2 - \| B^H_t - \theta_t \|_{\mathbb{R}^d}^2 \right) dt \right] = E_\theta \left[ \int_0^T \| g(B^H_t, t) \|_{\mathbb{R}^d}^2 + 2 \sum_{i=1}^d g_i(B^H_t, t)(B^H_t - \theta_t) dt \right],
\]

In addition,

\[
E_\theta \int_0^T \sum_{i=1}^d \left( g_i(B^H_t, t)(B^H_t - \theta_t) \right) dt
\]

\[
= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} g_i(x^1, \ldots, x^d, t)(x^i - \theta_t) \right.
\]

\[
\times e^{-\frac{\sum_{i=1}^d (x^i - \theta_t)^2}{2t^{2H}}} dx^1 \ldots dx^d \right) dt
\]

\[
= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g_i(x^1, \ldots, x^d, t) \right.
\]

\[
\times e^{-\frac{\sum_{i=1}^d (x^i - \theta_t)^2}{2t^{2H}}} dx^1 \ldots dx^d \right) dt
\]

\[
= \sum_{i=1}^d \int_0^T \left( 2^{2H} E_\theta \partial_i g_i(B^H_t, t) \right) dt = E_\theta \left[ \sum_{i=1}^d \int_0^T \left( 2^{2H} \partial_i g_i(B^H_t, t) \right) dt \right].
\]

Consequently, the risk difference equals

\[
\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( \| g(B^H_t, t) \|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g_i(B^H_t, t) \right) dt \right]. \quad (4.11)
\]
We can now state the following theorem.

**Theorem 2** Let $g: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $(B^H_t + g(B^H_t, t))_{t \in [0,T]}$ to dominate $B^H$ under the usual quadratic risk is

$$E_\theta \left[ \int_0^T \left( \|g(B^H_t, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g_i(B^H_t, t) \right) dt \right] < 0.$$ 

As an application, take $g$ of the form

$$g(x, t) = a t^{2H} \frac{\|x\|^2}{\|x\|^2} x, \quad (4.12)$$

where $a$ is a constant strictly positive and $r: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is bounded derivable function.

The next lemma gives a sufficient condition for $g$ in (4.12) to satisfy the assumption (A).

**Lemma 1** If $d \geq 3$ and $H < \frac{1}{2}$ then

$$E \left[ \int_0^T \frac{1}{\|B^H_t\|^2} dt \right] < \infty. \quad (4.13)$$

**Proof:** Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)((t, w); B^H_t(w) = 0) = 0$$

where $(dt \times P)$ is the product measure.

Using the change of variable and $d \geq 3$ we see that

$$E \int_0^T \frac{1}{\|B^H_t\|^2} dt = T \int_0^T \frac{dt}{t^{2H}} \int_{\mathbb{R}^d} \frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi} \|y\|^2} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,$$

where $C$ is a constant depending only on $d$. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

**Theorem 3** Assume that $d \geq 3$. If the function $r$, the constant $a$ and the parameter $H$ satisfy:

i) $0 \leq r(\cdot) \leq 1$

ii) $r(\cdot)$ is differentiable and increasing
iii) $0 < a \leq 2(d - 2)$ and $H < 1/2$, then the estimator
\[
\delta(B^H) = B^H_t - at^2H r\left(\|B^H_t\|^2\right) B^H_t, \quad t \in [0, T].
\]
dominates $B^H$.

**Proof:** It suffices to prove that $\Delta R(\theta) < 0$. From (4.11) and the hypothesis i) and ii) we can write
\[
\Delta R(\theta) = aE_\theta \left[ \int_0^T t^{2H} \left( \frac{ar^2(\|B^H_t\|^2)}{\|B^H_t\|^2} - 2(d - 2) \frac{r(\|B^H_t\|^2)}{\|B^H_t\|^2} \right)
\]
\[
- 4r'(\|B^H_t\|^2) \right) dt \right]
\]
\[
\leq a \left[ a - 2(d - 2) \right] E_\theta \left[ a \int_0^T t^{2H} \frac{r(\|B^H_t\|^2)}{\|B^H_t\|^2} \right].
\]
Combining this fact with the assumption iii) yields that the risk difference is negative. Which proves the desired result.

For $r = 1$, we obtain a James-Stein type estimator:

**Corollary 2** Let $d \geq 3$, $0 < H < \frac{1}{2}$ and $0 < a \leq 2(d - 2)$. Then the estimator
\[
\left( 1 - \frac{at^2H}{\|B^H_t\|^2} \right) B^H_t, \quad t \in [0, T]
\]
dominates $B^H$.

**Acknowledgement**
The authors would like to thank the editor Hira Koul and referees for several helpful corrections and suggestions that led to many improvements in the paper.

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