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Estimation of the drift of fractional Brownian motion

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Abstract
We consider the problem of efficient estimation for the drift of fractional Brownian motion $B^H := (B^H_t)_{t \in [0,T]}$ with hurst parameter $H$ less than $\frac{1}{2}$. We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

Key words : Fractional Brownian Motion, Stein estimate, MLE

2000 Mathematics Subject Classification: 60G15, 62G05, 62B05, 62M09.

1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \{ (B^H_1, ..., B^H_d); t \in [0,T] \}$ be a $d$-dimensional fractional Brownian motion (fBm) defined on the probability space $(\Omega, F, P)$. That is, $B^H$ is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for every $i = 1,...,d$ $B^{H,i}$ is a Gaussian process and covariance function given by

$$E(B^H_s, B^H_t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right), \quad s,t \in [0,T].$$

For each $i = 1, \ldots, d$, $(\mathcal{F}_t)_{t \in [0,T]}$ denotes the filtration generated by $\left( B^H_t \right)_{t \in [0,T]}$.

The fBm was first introduced by [3] and studied by [4]. Notice that if $H = \frac{1}{2}$, the
process $B^H_t$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let $M$ be a subspace of the Cameron-Martin space defined by

$$M = \left\{ \varphi : [0, T] \to \mathbb{R}^d; \varphi_t^i = \int_0^t \dot{\varphi}^i_s ds \text{ with } \varphi^i \in L^2([0, T]) \right\}.$$

and

$$\varphi^i \in L^{H+\frac{1}{2}}_2([0, T]), i = 1, ..., d.$$

Let $\theta = \{(\theta_1^i, ..., \theta_d^i); t \in [0, T]\}$ be a function belonging to $M$. Then, Applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to $P$ under which the process $\tilde{B}^H_t$ defined by

$$\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0, T] \tag{1.1}$$

is a fBm with Hurst parameter $H$ and mean zero. In this case, we say that, under the probability $P_\theta$, the process $B^H_t$ is a fBm with drift $\theta$.

We consider in this paper the problem of estimating the drift $\theta$ of $B^H_t$ under the probability $P_\theta$, with hurst parameter $H < 1/2$. We wish to estimate $\theta$ under the usual quadratic risk, that is defined for any estimator $\delta$ of $\theta$ by

$$R(\theta, \delta) = E_\theta \left[ \int_0^T |\delta_t - \theta_t|^2 dt \right]$$

where $E_\theta$ is the expectation with respect to a probability $P_\theta$.

Let $X = (X^1, ..., X^d)$ be a normal vector with mean $\theta = (\theta^1, ..., \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of $\theta$ is $X$ itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by $X$. That is

$$\sigma^2 d = E \left[ ||X - \theta||^2_d \right] = \inf_{\xi \in \mathcal{S}} E \left[ ||\xi - \theta||^2_d \right],$$

where $\mathcal{S}$ is the class of unbiased estimators of $\theta$ and $||.||_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

[12] constructed biased superefficient estimators of $\theta$ of the form

$$\delta_{n,b}(X) = \left( 1 - \frac{b}{a + ||X||^2} \right) X$$
for a sufficiently small and b sufficiently large when \( d \geq 3 \). \( \Box \) sharpened later this result and presented an explicit class of biased superefficient estimators of the form

\[
\left(1 - \frac{a}{||X||^2_d}\right)X, \text{ for } 0 < a < 2(d - 2).
\]

Recently, an infinite-dimensional extension of this result has been given by \( \Box \). The authors constructed unbiased estimators of the drift \( (\theta_t)_{t \in [0, T]} \) of a continuous Gaussian martingale \( (X_t)_{t \in [0, T]} \) with quadratic variation \( \sigma^2 \) \( dt \), where \( \sigma \in L^2([0, T], dt) \) is an a.e. non-vanishing function. More precisely, they proved that \( \hat{\theta} = (X_t)_{t \in [0, T]} \) is an efficient estimator of \( (\theta_t)_{t \in [0, T]} \). On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian process of the form:

\[
X_t := \int_0^t K(t, s)dW_s, \quad t \in [0, T],
\]

where \((W_t)_{t \in [0, T]}\) is a standard Brownian motion and \(K(\ldots)\) is a deterministic kernel. These estimators are biased and of the form \(X_t + D_t \log F\), where \(F\) is a positive superharmonic random variable and \(D\) is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that \( \hat{\theta} = B^H \) is an efficient estimator of \( \theta \) under the probability \( P_{\theta} \) with risk

\[
R(\theta, B^H) = E_{\theta} \left[ \int_0^T ||B^H_t - \theta_t||^2 dt \right] = \frac{T^{2H+1}}{2H+1} d.
\]

Moreover, we will establish that \( \hat{\theta} = B^H \) is a maximum likelihood estimator of \( \theta \).

In Section 4, we construct a class of biased estimators of James-Stein type of the form

\[
\delta(B^H)_t = \left(1 - a t^{2H} \left( \frac{r(||B^H||^2_t)}{||B^H||^2_t} \right) \right)B^H_t, \quad t \in [0, T].
\]

We give sufficient conditions on the function \(r\) and on the constant \(a\) in order that \(\delta(B^H)\) dominates \(B^H\) under the usual quadratic risk i.e.

\[
R(\theta, \delta(B^H)) < R(\theta, B^H) \text{ for all } \theta \in M. \quad (1.2)
\]

### 2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.
The fractional Brownian motion \( B^H_t \) has the following stochastic integral representation (see for instance, [1], [8])

\[
B^H_{i,t} = \int_0^t K_H(t, s) dW^i_s, \quad i = 1, ..., d; \quad t \in [0, T] \tag{2.3}
\]

where \( W = (W^1, ..., W^d) \) denotes the d-dimensional Brownian motion and the kernel \( K_H(t, s) \) is equal to

\[
c_H(t-s)^{H-\frac{1}{2}} + c_H(\frac{1}{2} - H) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2} - H}\right) \, du \quad \text{if } H \leq \frac{1}{2}
\]

\[
c_H(H-\frac{1}{2}) \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} \, du \quad \text{if } H > \frac{1}{2},
\]

if \( s < t \) and \( K_H(t, s) = 0 \) if \( s \geq t \). Here \( c_H \) is the normalizing constant

\[
c_H = \sqrt{\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma\left(2 - 2H\right)}}
\]

where \( \Gamma \) is the Euler function.

We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [11].

The left fractional Riemann-Liouville integral of \( f \in L^1((a,b)) \) of order \( \alpha > 0 \) on \( (a,b) \) is given at almost all \( x \in (a,b) \) by

\[
\Gamma^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) \, dy.
\]

If \( f \in \mathcal{I}^\alpha_{a+} (L^p(a,b)) \) with \( 0 < \alpha < 1 \) and \( p > 1 \) then the left-sided Riemann-Liouville derivative of \( f \) of order \( \alpha \) defined by

\[
D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} \, dy\right)
\]

for almost all \( x \in (a,b) \).

For \( H \in (0, 1) \), the integral transform

\[
(K_H f)(t) = \int_0^t K_H(t, s) f(s) \, ds
\]
is a isomorphism from \( L^2([0,1]) \) onto \( H^{1/2+}_{0+} (L^2([0,1])) \) and its inverse operator \( K^{-1}_H \) is given by

\[
K^{-1}_H f = t^{H-1/2} D^{H}_{0+} t^{1/2-H} f' \quad \text{for } H > 1/2, \quad (2.4)
\]

\[
K^{-1}_H f = t^{1/2-H} D^{H}_{0+} t^{H-1/2} D^{2H}_{0+} f' \quad \text{for } H < 1/2. \quad (2.5)
\]

Moreover, for \( H < \frac{1}{2} \), if \( f \) is an absolutely continuous function then \( K^{-1}_H f \) can be represented of the form (see [3])

\[
K^{-1}_H f = t^{H-1/2} D^{1/2-H}_{0+} t^{1/2-H} f'. \quad (2.6)
\]

### 3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function \( \theta = (\theta^1, \ldots, \theta^d) \) belonging to \( M \). An estimator \( \xi = (\xi^1, \ldots, \xi^d) \) of \( \theta = (\theta^1, \ldots, \theta^d) \) is called unbiased if, for every \( t \in [0,T] \)

\[
E_\theta(\xi^i_t) = \theta^i_t, \quad i = 1, \ldots, d
\]

and it is called adapted if, for each \( i = 1, \ldots, d \), \( \xi^i_t \) is adapted to \( (\mathcal{F}^i_t)_{t \in [0,T]} \).

Since for any \( i = 1, \ldots, d \), the function \( \theta^i_t \) is deterministic and

\[
\int_0^T (K^{-1}_H (\theta^i(s))^2 ds < \infty,
\]

then Girsanov theorem yields that there exists a probability measure \( P_\theta \) absolutely continuous with respect to \( P \) under which the process \( \tilde{B}^H := (\tilde{B}^H_t; t \in [0,T]) \) defined by

\[
\tilde{B}^H_t = B^H_t - \theta_t, \quad t \in [0,T] \quad (3.7)
\]

is a \( d \)-dimensional fBm with Hurst parameter \( H \) and mean zero. Moreover the Girsanov density of \( P_\theta \) with respect to \( P \) is given by:

\[
\frac{dP_\theta}{dP} = \exp \left[ \sum_{i=1}^d \left( \int_0^T K^{-1}_H (\theta^i(s))^2 ds \right) \right]
\]
and
\[ \tilde{B}_t^H = \int_0^t K_H(t, s) \, d\tilde{W}_s \]
where \( \tilde{W} \) is a d-dimensional Brownian motion under the probability \( P_\theta \) and
\[ \tilde{W}_i^t = W_i^t - \int_0^t K_H^{-1}(\theta^i)(s) \, ds, \quad i = 1, \ldots, d; \quad t \in [0, T]. \]
The equation (3.7) implies that \( B^H \) is an unbiased and adapted estimator of \( \theta \) under probability \( P_\theta \). In addition, we obtain the Cramer-Rao type bound:
\[ R(H, \hat{\theta}) := R(\theta, B^H) = \int_0^T E_{\theta} \| \tilde{B}_t^H \|^2 \, dt = d \int_0^T t^{2H} \, dt = \frac{T^{2H+1}}{2H + 1} d. \]
The first main result of this section is given by the following proposition which asserts that \( \hat{\theta} = B^H \) is an efficient estimator of \( \theta \).

**Theorem 1** Assume that \( H < \frac{1}{2} \). If \( \xi \) is an unbiased and adapted estimator of \( \theta \), then
\[ E_{\theta} \int_0^T \| \xi_t - \theta_t \|^2 \, dt \geq R(H, \hat{\theta}). \quad (3.8) \]

**Proof:** Since \( \xi \) is unbiased, then for every \( \varphi \in M \) we have
\[ E_{\varphi}(\xi^j_1) = E_{\varphi}(\varphi^j_1), \quad j = 1, \ldots, d. \]
Let \( \varphi = \theta + \varepsilon \psi \) with \( \psi \in M \) and \( \varepsilon \in \mathbb{R} \). Then for every \( t \in [0, T] \) and \( j \in \{1, \ldots, d\} \), we have
\[
E_{\theta + \varepsilon \psi}(\xi^j_1) &= E_{\theta + \varepsilon \psi}(\theta^j_1 + \varepsilon \psi^j_1) \\
&= E_{\theta + \varepsilon \psi}(\theta^j_1) + \varepsilon \psi^j_1.
\]
This implies that for every $j = 1, \ldots, d$
\[ \psi_t^j = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} E_{\theta + \varepsilon \psi} (\xi_t^j - \theta_t^j) \]
\[ = E \left( \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \exp \left[ \sum_{i=1}^{d} \left( \int_0^t K_H^{-1}(\theta^i + \varepsilon \psi^i)(s)dW_s^i \right) - \frac{1}{2} \int_0^t (K_H^{-1}(\theta^i + \varepsilon \psi^i)(s))^2 ds \right] (\xi_t^j - \theta_t^j) \right) \]
\[ = E_\theta \left( \sum_{i=1}^{d} \left[ \int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\theta^i)(s)ds \right] \right) \]
\[ = E_\theta \left( \sum_{i=1}^{d} \left[ \int_0^t K_H^{-1}(\psi^i)(s)d\tilde{W}_s^i \right] (\xi_t^j - \theta_t^j) \right) \]
\[ = E_\theta \left( \left[ \int_0^t K_H^{-1}(\psi^i)(s)d\tilde{W}_s^j \right] (\xi_t^j - \theta_t^j) \right). \]

Applying Cauchy-Schwarz inequality in $L^2(\Omega, dP_\theta)$, we obtain that for every $t \in [0, T]$
\[ ||\psi_t||^2 = \sum_{j=1}^{d} (\psi_t^j)^2 \leq \sum_{j=1}^{d} E_\theta \left( (\xi_t^j - \theta_t^j)^2 \right) E_\theta \left( \left[ \int_0^t K_H^{-1}(\psi^j)(s)d\tilde{W}_s^j \right]^2 \right) \]
\[ = \sum_{j=1}^{d} E_\theta \left[ (\xi_t^j - \theta_t^j)^2 \int_0^t (K_H^{-1}(\psi^j)(s))^2 ds \right]. \]

We take for each $j = 1, \ldots, d$, $\psi_t^j = t^{2H}$. Since $t \rightarrow t^{2H}$ is absolutely continuous function, then by (2.6), a simple calculation shows that
\[ K_H^{-1}(t^{2H}) = 2Ht^{H-\frac{1}{2}} \int_0^t s^{\frac{1}{2}-H} t^{H-\frac{1}{2}} ds \]
\[ = 2H \beta(\frac{1}{2} + H, H + \frac{1}{2}) t^{H-1/2} \]
\[ = 2H (\frac{1}{2} + H) t^{H-1/2}. \]

It is known that
\[ 0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1, 2]. \] (3.9)
Combining the facts that \( z \Gamma(z) = \Gamma(z + 1) \), \( z > 0 \), \( 2H \leq (H + \frac{1}{2})^2 \) and \((3.9)\), we obtain

\[
dt^{2H} = ||\psi_t||^2 \leq (\Gamma(\frac{3}{2} + H))^2 E_\theta(\||\xi_t - \theta_t||^2) \leq E_\theta(\||\xi_t - \theta_t||^2).
\]

Hence, by an integration with respect to \( dt \), we get

\[
R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T ||\xi_t - \theta_t||^2 dt.
\]

Therefore \((3.8)\) is satisfied.

**Corollary 1** The process \( \hat{\theta} = B^H \) is a maximum likelihood estimator of \( \theta \).

**Proof:** We have for every \( \psi \in M \)

\[
\frac{d}{d\varepsilon |_{\varepsilon = 0}} \exp \left[ \sum_{i=1}^d \int_0^t K^{-1}_H(\hat{\theta}^i + \varepsilon \psi^i)(s) dW^i_s - \frac{1}{2} \int_0^t (K^{-1}_H(\hat{\theta}^i + \varepsilon \psi^i)(s))^2 ds \right] = 0.
\]

Hence

\[
\sum_{i=1}^d \left( \int_0^t K^{-1}_H(\psi^i)(s) dW^i_s - \int_0^t K^{-1}_H(\psi^i)(s)K^{-1}_H(\hat{\theta}^i)(s) ds \right) = 0.
\]

Which implies that for every \( i = 1, ..., d \)

\[
E \left( \int_0^t K^{-1}_H(\psi^i)(s) dW^i_s - \int_0^t K^{-1}_H(\psi^i)(s)K^{-1}_H(\hat{\theta}^i)(s) ds \right)^2 = 0.
\]

Then, for each \( i = 1, ..., d \)

\[
W^i_t = \int_0^t K^{-1}_H(\hat{\theta}^i)(s) ds, \quad t \in [0, T].
\]

Therefore by \((2.3)\), we obtain that \( B^H = \hat{\theta} \).

**4 Superefficient James-Stein type estimators**

The aim of this section is to construct superefficient estimators of \( \theta \) which dominate, under the usual quadratic risk, the natural MLE estimator \( B^H \). The class of estimators considered here are of the form

\[
\delta(B^H)_t = B^H_t + g(B^H_t, t), \quad t \in [0, T]
\]

(4.10)
where \( g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) is a function. The problem turns to find sufficient conditions on \( g \) such that \( \mathcal{R}(\theta, \delta(B^H)) < \infty \) and the risk difference is negative, i.e.

\[
\Delta \mathcal{R}(\theta) = \mathcal{R}(\theta, \delta(B^H)) - \mathcal{R}(\theta, B^H) < 0.
\]

In the sequel we assume that the function \( g \) satisfies the following assumption:

\[
(A) \begin{cases} 
E_\theta \left[ \int_0^T \|g(B^H_t, t)\|^2 dt \right] < \infty, \\
the partial derivatives \( \partial_i g^i := \frac{\partial g}{\partial x^i}, \; i = 1, \ldots, n \) of \( g \) exist.
\end{cases}
\]

Then \( \mathcal{R}(\theta, \delta(B^H)) < \infty \). Moreover

\[
\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \|B^H_t + g(B^H_t, t) - \theta_t\|^2 dt \right] - E_\theta \left[ \int_0^T \|B^H_t - \theta_t\|^2 dt \right].
\]

In addition,

\[
E_\theta \int_0^T \sum_{i=1}^d \left( g^i(B^H_t, t)(B^H_t - \theta_t) \right) dt = \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t) \right) dt
\]

\[
= \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left( \int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t) \right) dt
\]

Consequently, the risk difference equals

\[
\Delta \mathcal{R}(\theta) = E_\theta \left[ \int_0^T \left( \|g(B^H_t, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B^H_t, t) \right) dt \right].
\]

(4.11)
We can now state the following theorem.

**Theorem 2** Let $g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $(B^H_t + g(B^H_t, t))_{t \in [0, T]}$ to dominate $B^H$ under the usual quadratic risk is

$$
E_\theta \left[ \int_0^T \left( \|g(B^H_t, t)\|^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B^H_t, t) \right) dt \right] < 0.
$$

As an application, take $g$ of the form

$$
g(x, t) = a t^{2H} \frac{\|x\|^2}{\|x\|^2} x,
$$

where $a$ is a constant strictly positive and $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is bounded derivable function.

The next lemma give a sufficient condition for $g$ in (4.12) to satisfies the assumption (A).

**Lemma 1** If $d \geq 3$ and $H < \frac{1}{2}$ then

$$
E \left[ \int_0^T \frac{1}{\|B^H_t\|^2} dt \right] < \infty.
$$

**Proof:** Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)((t, w); B^H_t(w) = 0) = 0$$

where $(dt \times P)$ is the product measure.

Using the change of variable and $d \geq 3$ we see that

$$
E \int_0^T \frac{1}{\|B^H_t\|^2} dt = \int_0^T \frac{d t}{t^{2H}} \int_{\mathbb{R}^d} \frac{e^{-\|y\|^2/2}}{\sqrt{2\pi} \|y\|^2} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,
$$

where $C$ is a constant depending only on $d$. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

**Theorem 3** Assume that $d \geq 3$. If the function $r$, the constant $a$ and the parameter $H$ satisfy:

i) $0 \leq r(.) \leq 1$

ii) $r(.)$ is differentiable and increasing
\( \text{iii) } 0 < a \leq 2(d - 2) \text{ and } H < 1/2, \)

then the estimator

\[
\delta(B^H_t) = B^H_t - at^{2H} \frac{r(\|B^H_t\|^2)}{\|B^H_t\|^2} B^H_t, \quad t \in [0, T].
\]

dominates \( B^H_t \).

**Proof:** It suffices to prove that \( \Delta R(\theta) < 0 \). From (4.11) and the hypothesis i) and ii) we can write

\[
\Delta R(\theta) = aE_{\theta} \left[ \int_0^T t^{4H} \left( \frac{ar^2(\|B^H_t\|^2)}{\|B^H_t\|^2} - 2(d - 2) \frac{r(\|B^H_t\|^2)}{\|B^H_t\|^2} \right. \right.
\]

\[
\left. \left. \left. \left. \left. - 4r'(\|B^H_t\|^2) \right) dt \right] \right] \right] \right]
\]

\[
\leq a \left[ a - 2(d - 2) \right] E_{\theta} \left[ a \int_0^T t^{4H} \frac{r(\|B^H_t\|^2)}{\|B^H_t\|^2} \right].
\]

Combining this fact with the assumption iii) yields that the risk difference is negative.

Which proves the desired result.

For \( r = 1 \), we obtain a James-Stein type estimator:

**Corollary 2** Let \( d \geq 3, 0 < H < \frac{1}{2} \) and \( 0 < a \leq 2(d - 2) \). Then the estimator

\[
\left( 1 - \frac{at^{2H}}{\|B^H_t\|^2} \right) B^H_t, \quad t \in [0, T]
\]

dominates \( B^H_t \).

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