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Structured portfolio analysis under SharpeOmega ratio

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Abstract

This paper deals with performance measurement of financial structured products. For this purpose, we introduce the SharpeOmega ratio, based on put as downside risk measure. This allows to take account of the asymmetry of the return probability distribution. We provide general results about the optimization of some standard structured portfolios with respect to the SharpeOmega ratio. We determine in particular the optimal combination of risk free, stock and call/put instruments with respect to this performance measure. We show that, contrary to Sharpe ratio maximization (Goetzmann *et al.,* 2002), the payoff of the optimal structured portfolio is not necessarily increasing and concave. We also discuss about the interest of the asset management industry to reward high Sharpe Omega ratios.

Key words: Structured portfolio, Performance measure, SharpeOmega ratio.

JEL Classification: C 61, G 11.

1 Introduction

Structured investments have been initially introduced by firms that searched for cheaper issue debt. For instance, convertible bonds can be sold instead of standard bonds to allow the conversion to equity. Structured products have been further extended to combinations of derivatives and financial instruments in order to provide funds with better risk/return profiles that are not always directly available on the financial market. They have became rather popular in the US in the 1980s and further introduced in Europe since the mid-1990s. These products are introduced to provide investors with highly targeted investments related to their performance objectives and risk profiles. They are created to satisfy specific needs that cannot be provided by standardized financial instruments, usually available in the financial markets. One of their main characteristics

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is fixed maturity. They are based on combinations of financial assets such as bonds, shares or indices, commodities,...and derivatives. The derivative component is often an option (Put or Call) chosen to provide some specific portfolio profile at maturity, while the other component is generally a note that delivers interest payments. Some structured products are linked to portfolio insurance. They guarantee a predetermined amount at maturity (usually a fixed percentage of the initial investment) whatever market fluctuations.

Structured products allow complex positions in options without the need for access to option market. There exists a large variety of such products, since a large number of underlyings and options can be introduced. The main classes are asset-linked notes and equity-linked notes and deposits, where the financial asset may be interest rate, equity, hybrid product, credit product, FX and commodity...(see Das (2000) for classifications of structured products). These products can incorporate plain-vanilla options 'corridor, turbo...) or exotic options such as barrier and rainbow products. The value of options, swaps...is determined from underlying asset prices. However, some mispricing may occur. Chen and Kensinger (1990) have examined Market-Index-Certificates of Deposit (MICD) in the US market, during a period of two months in 1988 and 1989. Using a comparison of the implied volatility of the S&P 500 option with the implied volatility of the MICDs option components, they have shown that significant differences exist between theoretical and market values. Chen and Sears (1990) also illustrate this feature for the S&P 500 Index Note (SPIN). Wasserfallen and Schenk (1996) has been led to the same conclusion when examining the pricing of capital-protected products issued in 1991/1992 in the Swiss market. Wilkens *et al.* (2003) have analyzed the German market through a large data set of classic structured products, traded in November 2001. They find "evidence of an overpricing of structured products, which can mostly be interpreted as in favor of the issuing institution." Stoimenov and Wilkens (2005) have also studied the German structured products, including in particular implicit exotic option components such as barrier and rainbow options. Their results suggest that "all types of equity-linked structured products are, on average, priced above their theoretical values and thus favor the issuing institution."

Main potential benefits of structured products are guarantees according to the kind of structured product and enhanced returns depending on portfolio profiles. The search for "optimal" structured products has been previously examined both from the insurance portfolio and the financial optimal positioning point of views. Leland and Rubinstein (1976) have introduced the option based portfolio insurance (OBPI). It consists of a portfolio invested in a risky underlying asset S (usually a financial index such as the $S\&P$) covered by a listed put written on it. Whatever the value of S at maturity T, the portfolio value will be always higher than the strike K of the put. At maturity, the investor can limit downside risk while participating in upside markets. Their optimal profiles can be determined from the optimal positioning problem which has been addressed in the partial equilibrium framework by Brennan and Solanki (1981) and by Leland (1980). The portfolio value is a function of the benchmark, in

a one period set up. Then, the optimal payoff which maximizes the expected utility is determined. It depends crucially on the risk aversion of the investor. Following this approach, Carr and Madan (2001) introduce markets in which exist out-of-the-money European puts and calls of all strikes. This assumption allows to examine the optimal positioning in a complete market and is the counterpart of the assumption of continuous trading. This approximation is justified when there is a large number of option strikes (eg. for the S&P500, for example). More specific insurance constraints can be considered and utility maximization can be solved (see e.g. Bertrand *et al.* (2001), El Karoui *et al.* (2005) and Prigent (2006) for quite general insurance constraints). The optimal positioning can be also examined within rank dependent expected utilities (RDEU) as in Jin and Zhou (2008) for the dynamic case and Prigent (2008) for the static case. In this framework, the choice of the threshold can be further examined (see Pfiffelmann, 2005; Pfiffelmann and Roger, 2008; Roger, 2008). Hens and Riger (2008) examine various features of structured products from the customer's perspective.

But structured products may also be illiquid, may include credit risk and be not daily priced. Additionally, they are often quite complex and their performance and risk evaluations are note easy to handle. Payoffs of structured products are non linear with respect to the underlying asset. This feature implies asymmetric return distributions. Their risks are similar to those of options and their return distributions are far from being lognormal. Therefore, we must be careful when evaluating their risks and performances. We must search for new performance measures, alternative to standard Sharpe ratio or Jensen alpha, to overcome shortcomings of performance measures based only on the first moments of the return distributions. Such performance measures are usually defined as reward-to-risk measures but, contrary for example to the Sharpe ratio, the risk measure is downside and aims to take the whole return distribution into account (see e.g. Pedersen and Satchell, 1998; Artzner *et al.*, 1999; Szegö, 2002). Keating and Shadwick (2002) have introduced such kind of risk measure to define a new performance measure based on a gain-loss approach. This one is called the Omega measure. It takes account of investor loss aversion, which is in line with results of Tversky and Kahneman (1992). It has been applied in finance to examine or instance equities or hedge funds. The Omega measure is equal to the ratio of the expected gains and the expected losses, defined with respect to a given threshold. As noted by Kazemi *et al.* (2004), it corresponds to the ratio of the expectations of a call option divided by a put option written on the underlying asset. The strike price is the given threshold. The SharpeOmega measure, introduced by Kazemi *et al.* (2004), is equal to the Omega measure minus 1. Such measure has been previously introduced to examine performance of some structured products, such as those related to portfolio insurance. Bertrand and Prigent (2008) use the Omega performance measure to compare standard portfolio insurance strategies. They show that the CPPI method provides better results than the OBPI one for "rational" thresholds. Non normal distributions can also be proposed to model structured product returns, for example Johnson

distributions. In this framework, Perez (2004) have used the Omega approach to test adequacy of these distributions. Passow (2005) provide explicit representations for Omega and SharpeOmega with all four Johnson distributions. Using a Hedge fund index as back-testing, he shows that Johnson-Omega provides significantly higher returns. Other researchers have focused on the problem of portfolio allocations in order to maximize Omega (see Avouyi-Dovi *et al.* 2004). Empirical results show that this measure is more stable than other risk measures such as RoCVaR , RoVaR and Sharpe (see Hentati *et al.* 2010) but it has many local solutions because of the non-convexity of Omega function. The resolution of the global optimum is proposed by Bartholomew-Biggs *et al.* (2009) by using a NAG library implementation of the Huyer & Neumaier MCS method. Based on another approach, Hentati and Prigent (2010) introduce Gaussian mixtures to model empirical distributions of financial assets and solve the portfolio optimization problem in a static way, taking account of discrete time portfolio rebalancing.

In this paper, we propose to analyze structured products by using the SharpeOmega ratio. It is well known that the Sharpe ratio can be manipulated by option-like strategies (see Henriksson and Merton, 1981; Dybvig and Ingersoll, 1982). In this context, Goetzmann *et al.* (2002) determine portfolio strategies which maximize the Sharpe ratio. They derive general conditions to achieve the maximum. They prove that appropriate combinations of puts and calls lead to significantly higher Sharpe ratios than "linear" portfolios. Our approach is quite similar, except that we use the SharpeOmega ratio instead of the Sharpe ratio itself. For this purpose, we consider a portfolio manager who invests in three assets: a free risk market account, denoted by B, a risky asset (equity), denoted by S and Call/Put written on this equity. Our aim is to maximize and analyze SharpeOmega ratio under given constraints. We begin by determining the necessary conditions to determine precisely the downside risk component. Subsequently, we study the minimization problem of the Put under the constraint of fixed expectation.

The paper is organized as follows. Section 2 recalls definitions and main properties of the Omega and Sharpe Omega measures. Section 3 deals with various portfolio optimizations with respect to these ratios. We prove that, unlike the result of Goetzmann *et al. (*2002) related to the Sharpe ratio maximization, the payoff of the optimal structured portfolio is not always increasing and concave. It can correspond for instance to a straddle. This result is in line with previous results about portfolio optimization within rank dependent utility, as in Prigent (2008).

2 Omega measure

The Omega measure is based on the portfolio return values below and above a given threshold. It is defined as the probability weighted ratio of gains to losses relative to a return threshold. The Omega measure is compatible with the second order stochastic dominance. This measure can potentially take account of the whole probability distribution of the returns. It requires no parametric assumption on the distribution and is equal to:

$$\Omega_L(X) = \frac{\int_L^b (1 - F(x))\, dx}{\int_a^L F(x)\, dx} = \frac{I_{L,2}(X)}{I_{L,1}(X)}, \tag{1}$$

where $F(.)$ is the cdf of the random variable X (for example equal to the portfolio return) defined on the interval $[a, b]$. The level L is the threshold chosen by the investor: returns smaller than L are viewed as losses (which correspond to $I_{L,1}(X)$) and those higher than L are gains (component $I_{L,2}(X)$). Thus, for a given threshold L, the investor would prefer the portfolio with the highest Omega measure.

As shown by Kazemi, Schneeweis and Gupta (2003), the Omega function is equal to:

$$\Omega_L(X) = \frac{\mathbb{E}_{\mathbb{P}}\left[(X - L)^+\right]}{\mathbb{E}_{\mathbb{P}}\left[(L - X)^+\right]}. \tag{2}$$

This is the ratio of the expectations of gains above the given level L upon the expectation of losses below . Therefore, $\Omega_F(L)$ can be interpreted as a ratio call/put defined on the same underlying asset X, with strike L and computed with respect to the historical probability $\mathbb{P}$. The put correspond to the risk measure component. It allows the control of the losses below the threshold L.

Note that Omega functions satisfies the following properties:

- For $L = \mathbb{E}_{\mathbb{P}}[X]$, $\Omega_L(X) = 1$.
- Ω is a monotone decreasing function with respect to the threshold.
- $\Omega_L(X) = \Omega_L(Y)$ for all thresholds L if and only X and Y have the same cdf ($F_X = F_Y$).

Kazemi *et al.* (2003) define the Sharpe Omega by:

$$\text{Sharpe-Omega} = \Omega_L(X) - 1 = \frac{\mathbb{E}_{\mathbb{P}}[X] - L}{\mathbb{E}_{\mathbb{P}}\left[(L - X)^+\right]}. \tag{3}$$

If $\mathbb{E}_{\mathbb{P}}[X] < L$, the Sharpe Omega will be negative otherwise it will be positive.Typically, consider the payoff X of a stock S at time T which is modelled by a geometric Brownian motion:

$$X = S_0 \exp[(\mu - \sigma^2/2)T + \sigma W_T],$$

where W_T has the Gaussian distribution $\mathcal{N}(0,T)$. Then, $\mathbb{E}_{\mathbb{P}}[X] = S_0 \exp[\mu T]$ does not depend on the volatility. Thus, if $S_0 \exp[\mu T] < L$ then the Sharpe Omega is an increasing function of the volatility (due to the Vega of the put). If $S_0 \exp[\mu T] > L$, the Sharpe Omega is a decreasing function of the volatility.

3 Maximizing the Sharpe Omega Ratio

In this section, we examine the Sharpe Omega maximization for various portfolios based on combinations of a risk free asset, a risky asset on some European options written on it. Denote by $V_T(\theta)$ the portfolio value corresponding to the weighting vector θ. We have to solve the following optimization problem:

$$\underset{(\theta)}{Max}\ S\Omega_L(V_T) = \underset{(\theta)}{Max}\ \frac{\left(\overline{V_T}(\theta) - L\right)}{\mathbb{E}\left[(L - V_T(\theta))^+\right]}. \tag{4}$$

The parameter L is a threshold satisfying $0 < L < \overline{V_T}$. We maximize this function under the budget constraint V_0.

We also impose that the portfolio value is positive, whatever the market evolutions.

3.1 Maximizing the Sharpe Omega Ratio in a complete market

In this section, we examine the Sharpe Omega maximization problem in a complete financial market. Recall that a discrete-time market can be complete if there exists sufficiently available options.

3.1.1 Maximizing the Sharpe ratio

Consider for instance a standard single-period model. Assume that there exists a finite set of random events $\{\omega_i,\ i = 1, ..., d\}$ endowed with a probability measure $\mathbb{P} = \{p_i,\ i = 1, ..., d\}$. The risk neutral probability is denoted by $\mathbb{Q} = \{q_i,\ i = 1, ..., d\}$.

For the Sharpe maximization (with $L = V_0 e^{rT}$), Goetzmann *et al.* (2002) take account of the budget constraint and fix the portfolio value expectation to M. Thus, they consider the following Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{d} v_i^2 p_i + \lambda \left(V_0 e^{rT} - L - \sum_{i=1}^{d} v_i q_i \right) + \mu \left(M - \sum_{i=1}^{d} v_i p_i \right). \tag{5}$$

Thus, the optimal solution exists and is given by:

$$v_i^* = M + \left(\frac{M}{\sum_{i=1}^{d} \frac{(q_i)^2}{p_i} - 1} \right) \left(1 - \frac{q_i}{p_i} \right).$$

3.1.2 Maximizing the Sharpe Omega ratio

We consider portfolios with excess returns $\widetilde{v}_i$ w.r.t. the threshold L (i.e. $\widetilde{v}_i = v_i - L$). The Sharpe Omega ratio of such portfolios is equal to:

$$S\Omega_L(V) = \frac{\sum_{i=1}^d \widetilde{v}_i p_i}{\sum_{i=1}^d [-\widetilde{v}_i]^+ p_i}.$$

Taking account of the budget constraint and fixing the portfolio value expectation to M, we consider the following Lagrangian: (notations: $\widetilde{V}_0 = V_0 e^{rT} - L$, $\widetilde{M} = M - L$)

$$\mathcal{L} = \sum_{i=1}^d [-\widetilde{v}_i]^+ p_i + \lambda \left(\widetilde{V}_0 - \sum_{i=1}^d \widetilde{v}_i q_i \right) + \mu \left(\widetilde{M} - \sum_{i=1}^d \widetilde{v}_i p_i \right). \tag{6}$$

We assume that $\forall i$, $\widetilde{v}_i \neq 0$. The first-order conditions for a minimun are:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \widetilde{v}_i} = -p_i I_{\{\widetilde{v}_i < 0\}} - \lambda q_i - \mu p_i, \\
0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = \widetilde{V}_0 - \sum_{i=1}^d \widetilde{v}_i q_i, \\
0 &= \frac{\partial \mathcal{L}}{\partial \mu} = \widetilde{M} - \sum_{i=1}^d \widetilde{v}_i p_i.
\end{aligned}$$

It implies that the ratio $\frac{q_i}{p_i}$ takes no more than two values. Thus, except very special cases, an interior solution does not exist.

Suppose that we search to maximize other Kappa measures than the Sharpe Omega ratio (see Kaplan and Knowles, 2004). These measures are defined by: for any non null integer n,

$$K_L(V) = \frac{\sum_{i=1}^d \widetilde{v}_i p_i}{\left(\sum_{i=1}^d [-\widetilde{v}_i]^{+n} p_i \right)^{\frac{1}{n}}}.$$

Note that for $n = 1$, we recover the Sharpe Omega ratio and, for $n = 2$, the Sortino ratio.

For fixed expectation, the maximization of the Kappa measure is equivalent to the minimization of the risk measure component. This latter one is also equivalent to the minimization of expression $\sum_{i=1}^d [-\widetilde{v}_i]^{+n} p_i$.

Thus, the Lagrangian is defined by:

$$\mathcal{L} = \sum_{i=1}^d [-\widetilde{v}_i]^{+n} p_i + \lambda \left(\widetilde{V}_0 - \sum_{i=1}^d \widetilde{v}_i q_i \right) + \mu \left(\widetilde{M} - \sum_{i=1}^d \widetilde{v}_i p_i \right). \tag{7}$$

We assume that $\forall i,\ \widetilde{v}_i \neq 0$. The first-order conditions for a minimum are:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \widetilde{v}_i} = -n\left[-\widetilde{v}_i\right]^{n-1} p_i I_{\{\widetilde{v}_i<0\}} - \lambda q_i - \mu p_i, \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = \widetilde{V}_0 - \sum_{i=1}^{d} \widetilde{v}_i q_i, \\ 0 &= \frac{\partial \mathcal{L}}{\partial \mu} = \widetilde{M} - \sum_{i=1}^{d} \widetilde{v}_i p_i. \end{aligned}$$

Therefore, even for $n > 1$, the existence of an interior solution that the ratio $\frac{q_i}{p_i}$ is constant for all random events ω_i such that $\widetilde{v}_i^* > 0$. Therefore, as for the Omega Sharpe ratio ($n = 1$), generally there exists no interior solution.

To get an interior solution, we have to add other constraints. For example, we can fix the variance, which leads to $\sum_{i=1}^{d} \widetilde{v}_i^2 p_i = s^2$. In this framework, the Lagrangian is defined by:

$$\mathcal{L} = \sum_{i=1}^{d} \left[-\widetilde{v}_i\right]^{+n} p_i + \lambda\left(\widetilde{V}_0 - \sum_{i=1}^{d} \widetilde{v}_i q_i\right) + \mu\left(\widetilde{M} - \sum_{i=1}^{d} \widetilde{v}_i p_i\right) + \nu\left(s^2 - \sum_{i=1}^{d} \widetilde{v}_i^2 p_i\right). \tag{8}$$

The first-order conditions for a minimum are now:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \widetilde{v}_i} = -n\left[-\widetilde{v}_i\right]^{n-1} p_i I_{\{\widetilde{v}_i<0\}} - \lambda q_i - \mu p_i - 2\nu \widetilde{v}_i p_i, \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = \widetilde{V}_0 - \sum_{i=1}^{d} \widetilde{v}_i q_i, \\ 0 &= \frac{\partial \mathcal{L}}{\partial \mu} = \widetilde{M} - \sum_{i=1}^{d} \widetilde{v}_i p_i. \\ 0 &= \frac{\partial \mathcal{L}}{\partial \nu} = s^2 - \sum_{i=1}^{d} \widetilde{v}_i^2 p_i. \end{aligned}$$

The optimal solution satisfies:

$$\begin{aligned} -n\left[-\widetilde{v}_i\right]^{n-1} - 2\nu\widetilde{v}_i &= \lambda\frac{q_i}{p_i} + \mu, \text{ if } \widetilde{v}_i < 0 \\ -2\nu\widetilde{v}_i &= \lambda\frac{q_i}{p_i} + \mu, \text{ if } \widetilde{v}_i > 0 \\ &\text{with} \\ \widetilde{V}_0 &= \sum_{i=1}^{d} \widetilde{v}_i q_i,\ \widetilde{M} = \sum_{i=1}^{d} \widetilde{v}_i p_i,\ s^2 = \sum_{i=1}^{d} \widetilde{v}_i^2 p_i. \end{aligned}$$

For $n = 1$, we get:

$$\begin{aligned}
\widetilde{v}_i &= \left(\frac{1}{-2\nu}\right)\left(\lambda\frac{q_i}{p_i} + [\mu + \mathbb{I}_{\widetilde{v}_i<0}]\right) \\
&\text{with} \\
(-2\nu)\widetilde{V}_0 &= \sum_{i=1}^{d}\left(\lambda\frac{q_i^2}{p_i} + [\mu + \mathbb{I}_{\widetilde{v}_i<0}]\, q_i\right), \\
(-2\nu)\ \widetilde{M} &= \sum_{i=1}^{d}\left(\lambda q_i + [\mu + \mathbb{I}_{\widetilde{v}_i<0}]\, p_i\right), \\
(-2\nu)^2\, s^2 &= \sum_{i=1}^{d}\left(\lambda\frac{q_i}{p_i} + [\mu + \mathbb{I}_{\widetilde{v}_i<0}]\right)^2 p_i.
\end{aligned}$$

Solving the system composed by the last three equations provides the value of the Lagrangian parameters λ, μ, and ν. Note that such result can be examined for instance for the standard Black-Scholes model. In that case, the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ has conditional expectations that we denote by the process η.

Therefore, an optimal interior solution can be expressed by:

$$\widetilde{V}^* = \left(\frac{1}{-2\nu}\right)\left(\lambda\eta + \left[\mu + \mathbb{I}_{\widetilde{V}^*<0}\right]\right).$$

Nevertheless, as it has been seen, except for additional constraints, the Sharpe Omega maximization has no interior solution. Therefore, we can search directly optimal solutions for special structured products, as in Goetzmann *et al.* (2002).

3.2 Maximizing the Sharpe Omega Ratio with a money market account, an equity and a put

In this section, we calculate the Sharpe-Omega ratio of a portfolio composed of a risk-free asset B,a risky asset S and a put on S. We assume a single period optimization problem with a static allocation (determined at the beginning of the period).

3.2.1 Portfolio value

We assume that the portfolio is composed of α money market account, denoted by B, β risky asset (an equity for example) and γ Put option. The time period is $[0, T]$. Thus, the portfolio value V_T is given at maturity T by:

$$V_T = \alpha B_T + \beta S_T + \gamma(K - S_T)^+, \tag{9}$$

where S_T is the value of the risky asset at maturity T and K is the strike price of Put.

The dynamics of the market value of the risky asset S are those of the standard geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{10}$$

where μ and σ are respectively the drift and the volatility of S_t, and W is a standard Brownian motion.

Thus we have:

$$S_T = S_0 \exp[(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T].$$

Then, the expected return expectation of asset S at time T is given by:

$$\mathbb{E}(S_T) = S_0 \exp[\mu T]. \tag{11}$$

The value of the riskless asset B evolves according to:

$$dB_t = B_t r dt. \tag{12}$$

In this context, the value of the initial investment amount is given by:

$$V_0 = \alpha B_0 + \beta S_0 + \gamma P_0(K), \tag{13}$$

where $P_0(K)$ is the Black and Scholes value of the Put option at time $t = 0$.

Then, the amount γ invested in the put can be written as follows:

$$\gamma = \frac{V_0 - \alpha B_0 - \beta S_0}{P_0(K)}. \tag{14}$$

3.2.2 Analysis of the Sharpe-Omega risk component

The risk component corresponds to the expectation $\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right]$. In what follows, we calculate $\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right]$.

Let f be the portfolio payoff. We have: $V_T = f(S_T)$. Then we get:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] &= \int [L - f(s)]^+ d\mathbb{P}_{S_T}(s), \\ &= \int_{f(S_T)\leq L} [L - f(s)] d\mathbb{P}_{S_T}(s).\end{aligned}$$

If the portfolio is the asset combination given in (13), than we have:

$$\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] = \int_{f(S_T)\leq L} \left(L - \alpha B_T - \beta s - \gamma(K - s)^+\right) d\mathbb{P}_{S_T}(s).$$

We begin by determining the integration domain for S_T, defined by set D.

$$D = \{s \,|\, f(s) \leq L\}.$$

Thus, according to the values of α, β and γ, f has different shapes. Due to the positivity constraint, the amounts must fulfill the following conditions:

- $\alpha B_T + \gamma K \geq 0$,
- $\alpha B_T + \beta K \geq 0$, and
- $\beta \geq 0$.

Two main cases have to be distinguished:

- $\beta < \gamma$: the function f is increasing, which is strictly concave (resp. strictly convex) if and only if $\gamma < 0$ (resp. $\gamma > 0$).
- $\beta > \gamma$: the function f is decreasing on $[0, K]$ then increasing on $[K, +\infty[$.

CASE 1 :$\beta > \gamma$

The set D is not empty if $L \geq \alpha B_T + \gamma K$. Thus, we have two sub-cases:

a.1 $L \leq \alpha B_T + \beta K$ then: $V_T \leq L$ is equivalent to $S_T \leq \frac{L - \alpha B_T - \gamma K}{\beta - \gamma}$;

a.2 $L > \alpha B_T + \beta K$ then: $V_T \leq L$ is equivalent to $S_T \leq \frac{L - \alpha B_T}{\beta}$.

In what follows, we set:

$$k_1 = \frac{L - \alpha B_T - \gamma K}{\beta - \gamma},$$

and

$$k_2 = \frac{L - \alpha B_T}{\beta}.$$

For case (a.1), $\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right]$ can be written as follows :

$$\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] = \int_0^{k_1} (L - \alpha B_T + (\gamma - \beta)\, s - \gamma K)\, d\mathbb{P}_{S_T}(s). \tag{15}$$

Then, we get the following relation:

$$\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] = (L - \alpha B_T - \gamma K) \int_0^{k_1} d\mathbb{P}_{S_T}(s) + (\gamma - \beta) \int_0^{k_1} s d\mathbb{P}_{S_T}(s). \tag{16}$$

Notations: $\overline{z} = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T$ and $\phi(.)$ is the cumulative distribution function of the standard Gaussian distribution.[1]

Then, we have:

$$\int_0^{k_1} d\mathbb{P}_{S_T}(s) = \phi\left(-\frac{\overline{z} - \ln k_1}{\sigma\sqrt{T}}\right),$$

[1] Recall that we assume that S is lognormally distributed.

and

$$\int_0^{k_1} s d\mathbb{P}_{S_T}(s) = \exp\left(\overline{z} + \frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z} - \ln k_1 + \sigma^2 T}{\sigma\sqrt{T}}\right).$$

Therefore, we deduce:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] &= (L - \alpha B_T - \gamma K)\left[\phi\left(-\frac{\overline{z} - \ln k_1}{\sigma\sqrt{T}}\right)\right] \\ &\quad + (\gamma - \beta)\exp\left(\overline{z} + \frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z} - \ln k_1 + \sigma^2 T}{\sigma\sqrt{T}}\right).\end{aligned} \tag{17}$$

Set:

$$A = \phi\left(-\frac{\overline{z} - \ln k_1}{\sigma\sqrt{T}}\right),$$

and

$$B = \phi\left(-\frac{\overline{z} - \ln k_1 + \sigma^2 T}{\sigma\sqrt{T}}\right).$$

Lemma 1 *If $\beta > \gamma$ and $\alpha B_T + \gamma K \leq L \leq \alpha B_T + \beta K$, the put component is given by:*

$$\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] = (L - \alpha B_T - \gamma K)A + (\gamma - \beta)\exp\left(\overline{z} + \frac{\sigma^2}{2}T\right)B.$$

Similarly, for case (a.2), we get the following result (see Appendix 1 for more details).

Lemma 2 *If $\beta > \gamma$ and $L > \alpha B_T + \beta K$, the put component is given by:*

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] &= -\gamma K\phi\left(-\frac{\overline{z} - \ln K}{\sigma\sqrt{T}}\right) + (L - \alpha B_T)\phi\left(-\frac{\overline{z} - \ln k_2}{\sigma\sqrt{T}}\right) \\ &\quad + \gamma\exp\left(\overline{z} + \frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z} - \ln K + \sigma^2 T}{\sigma\sqrt{T}}\right) \\ &\quad - \beta\exp\left(\overline{z} + \frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z} - \ln k_2 + \sigma^2 T}{\sigma\sqrt{T}}\right)\end{aligned}$$

CASE 2 :$\beta \leq \gamma$

Here, inequation $f(S_T) \leq L$ can have solutions only if $L \geq \alpha B_T + \beta K$. As previous case, two sub-cases are examined:

b.1 $L \leq \alpha B_T + \gamma K$. Then the portfolio value V_T is smaller than threshold L if and only if

$$k_1 \leq S_T \leq k_2;$$

b.2 $L > \alpha B_T + \gamma K$. Then the portfolio value V_T is smaller than threshold L if and only if

$$S_T \leq k_2.$$

Thus, we obtain the bounds of S_T according the the value of β et γ.
We perform the same calculation for the cases (b.1) and (b.2).
The first case gives the following value of $\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right]$:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] &= \int_{k_1}^{k_2} \left(L - \alpha B_T - \beta s - \gamma (K - s)^+\right) d\mathbb{P}_{S_T}(s) \\
&= \int_{k_1}^{K} (L - \alpha B_T + (\gamma - \beta) s - \gamma K)\, d\mathbb{P}_{S_T}(s) \\
&\quad + \int_{K}^{k_2} (L - \alpha B_T - \beta s)\, d\mathbb{P}_{S_T}(s)
\end{aligned}$$

Finally, in (b.2) case, we have:

$$\mathbb{E}_{\mathbb{P}}\left[(L - V_T)^+\right] =$$

$$\int_0^{k_2} (L - \alpha B_T - (\beta - \gamma) s - \gamma K)\, d\mathbb{P}_{S_T}(s) + \int_K^{k_2} (L - \alpha B_T - \beta s)\, d\mathbb{P}_{S_T}(s).$$

Details about calculation are provided in Appendix 1.

3.2.3 Conditions on portfolio weights

Assume that the expected value is fixed and equal to the level M. this condition yields to:

$$\alpha B_T + \beta S_0 e^{\mu T} + \gamma \mathbb{E}_{\mathbb{P}} [K - S_T]^+ = M > L, \tag{18}$$

where $\mathbb{E}_{\mathbb{P}} [K - S_T]^+$is like a Put "à la Black-Scholes" without the discount factor. We denote it by:

$$BS(\mu) e^{\mu T}. \tag{19}$$

In that case, the Sharpe Omega maximization is equivalent to minimization of the put component with a single variable. Indeed, both β and γ are determined from relations:

$$\begin{cases} \alpha B_0 + \beta S_0 + \gamma P_0(K) = V_0, \\ \beta S_0 e^{\mu T} + \gamma BS(\mu) e^{\mu T} = M - \alpha B_T. \end{cases} \tag{20}$$

Thus, β and γ can be expressed as function of α.
Introduce the following parameters:

$$a_{11} = \frac{-BS(\mu) e^{\mu T} B_0 + P_0(K) B_0 e^{rT}}{\Delta}, \tag{21}$$

$$b_{11} = \frac{BS(\mu) V_0 e^{\mu T} - P_0(K) M}{\Delta}. \tag{22}$$

and

$$a_{21} = \frac{S_0 B_0}{\Delta}\left(e^{\mu T} - e^{rT}\right), \tag{23}$$

$$b_{21} = \frac{S_0}{\Delta}\left(M - S_0 e^{\mu T} V_0\right). \tag{24}$$

Lemma 3 *For fixed portfolio value expectation, the amounts invested on the risky asset and the put are linear functions of the amount invested on the risk free asset. They are given by:*[2]

$$\beta = a_{11}\alpha + b_{11}, \tag{25}$$

$$\gamma = a_{21}\alpha + b_{21}. \tag{26}$$

Remark 4 *Parameters a_{11} and a_{21} are non positive. The function $b_{11}(M)$ is always increasing and $b_{21}(M)$ is decreasing (see Appendix 3 for proof).*

3.2.4 Constraints on portfolio optimization

The maximization problem is subject to the condition of positivity of V_T ($V_T \geqslant 0$). Therefore, the following constraints must be satisfied:

Positivity for $S_T = 0$: It corresponds to inequality $\alpha B_T + \gamma K \geq 0$. Thus, we obtain:

$$\alpha B_T + a_{21} \times K \times \alpha + K b_{21} \geq 0 \tag{27}$$

which is equivalent to the following constraint on amount α:

$$\alpha\left(-B_T - a_{21} \times K\right) \leq K b_{21}.$$

Positivity for $S_T = K$: This second constraint is written as follows:

$$\alpha B_T + \beta K \geq 0.$$

Therefore, we get:

$$\alpha B_T + a_{11} \times K \times \alpha + K b_{11} \geq 0, \tag{28}$$

which is equivalent to:

$$\alpha\left(-B_T - a_{11} \times K\right) \leq K b_{11}$$

Positivity for $S_T > K$: This third condition corresponds to $\beta \geq 0$. It is equivalent to $-a_{11}\alpha \leq b_{11}$.

Note that, for threshold L different from 0, the portfolio including the risk-free asset yields the highest Sharpe-Omega ratio (equal to $+\infty$). However, other constraints can be introduced, such as a minimal participation on risky asset fluctuations. For this purpose, we prevent the portfolio to be entirely invested on the risk-free asset ($\alpha B_0 \neq V_0$).

[2] See Appendix 2.

3.2.5 Shape of optimal portfolio values

We examine optimal portfolio payoffs according to various strike values. We consider the following parameters values:

$$S_0 = 100, B_0 = 1, \sigma = 0.15, \mu = 0.05, r = 0.03, T = 1, V_0 = 1000, L = V_0.$$

We search for the optimal amounts $(\alpha^*, \beta^*, \gamma^*)$ that maximize the Sharpe Omega ratio. For this purpose, first we determine the optimal solution for fixed expectation M. Then, in a second step, we search the optimal value α^* by varying the level of M.

We begin by examining the at-the-money case (the strike of the put is equal to the spot value $K = S_0 = 100$). Figure 1 illustrates relationship between the reward (the expected value M) and the risk (the put component $\mathbb{E}(L - V_T)^+$). For the efficient frontiers (Put , Expected return), maximization at given expected return M, the put price exhibits the following variations: it is decreasing when M varies between 1000 and $1025,4$ and increasing for M greater than $1025,4$. Figure 2 provides the optimal weights according to expected value M.

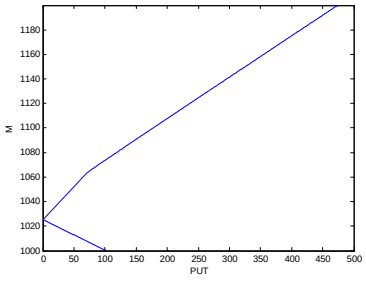


Fig.1. Efficient frontier
$(Put\ \mathbb{E}(L - V_T)^+; M)$

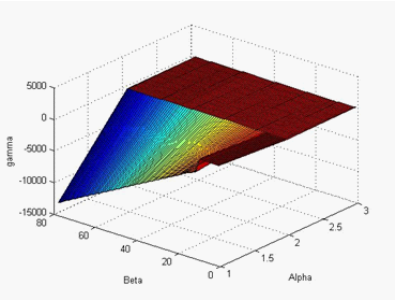


Fig.2. Optimal allocations
for different values of M

Figure 3 displays the Sharpe-Omega value of the optimal portfolio as function of the expected portfolio value. The maximum Sharpe-Omega ratio $S\Omega^*_{K=100}$ is equal to $86,88$ (cf. Figure 3) to be compared to $1,31$ for the risky asset. The maximum ratio is reached for $M^* = 1025,4$, which is smaller than $V_0 \exp^{\mu T}$. Thus, the value of M^* is relatively low. This is due to the fact that, without specific constraint on the amount invested on the risk free asset, the optimal would correspond to a whole investment on this risk-free asset. The analysis of optimal ratio sensitivity to the strike K is illustrated in Figure 4. The optimal ratio decreases for values of K varying from 90 to 100 and then increases for K below 90.

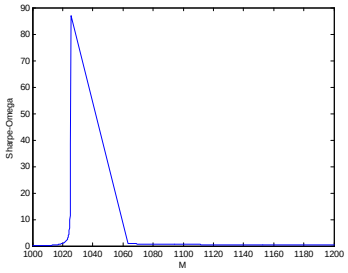


Fig.3. Sharpe-Omega ratio w.r.t. M

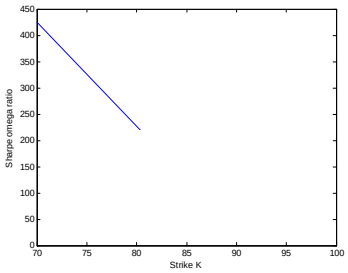


Fig.4. Sharpe-Omega ratio for K in [70,100]

Indeed, the optimal portfolio structure changes dramatically when K becomes slightly smaller than 100. Optimal portfolios when K is ranged between 90 and 100 have the same shape and do not contain the risky asset S (beta is zero). Nonzero allocation to risky asset appears for K below 90.

The optimal portfolio corresponding to $K = 100$ has an expected return of 2.54% with minimum value at the end of the period of 999.8 (0.02% of the initial value V_0).

As illustrated in Figure 5, the optimal payoff is similar to a naked put. The optimal allocation shows 97.6% invested in risk-free, 0% in risky asset and 3% used as a hedge. Therefore, the maximization of Sharpe-Omega ratio yields to an optimal portfolio which limits the downside risk and enhances the performance profile on the left side (i.e. when risky asset drops). Note that the shape of V_T^* is not always increasing and concave as proved by Goetzmann *et al.* (2002) for the Sharpe ratio maximization. When we focus on the impact of the μ value on the shape of the optimal payoff V_T, we obtain the same result concerning the optimal allocation (with $K = 100$).

The corresponding optimal portfolio profiles are displayed in Figure 6 for different values of the drift μ.

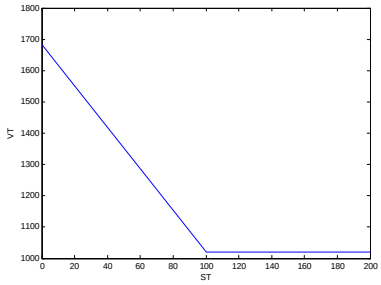


Fig.5. Optimal portfolio payoff for at-the-money put

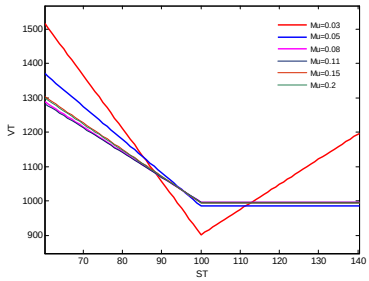


Fig.6. Optimal portfolio payoff for various μ

The portfolio payoff changes significantly when μ becomes higher than the risk-free rate ($r = 0.03$). Furthermore, optimal portfolio allocation is concentrated on risk-free and put instruments once μ is higher than 0.03.

μ	M^*	Put^*	$S\Omega^*$	$w_{\alpha^*}(\%)$	$w_{\beta^*}(\%)$	$w_{\gamma^*}(\%)$
0,05	1025,4	0,2924	86,88	96.99	0	3.01
0,08	1019	0,1682	112,96	97.02	0	2.98
0,11	1013,9	0,04887	284,44	97.03	0	2.97
0,15	1008,75	0,02716	322,19	97.04	0	2.96

Table 1: Optimal parameters for different values of μ

3.2.6 Portfolio excluding risk-free instrument

We run the same optimization as in previous section. Sharpe-Omega ratio is minimum for $K = 93$. It is interesting to note that the optimal ratio calculated for $K = 100$ is very close to value in the case of portfolio including risk-free instrument (2.74). Unlike the previous case, optimal ratio exceeds 2.74 only when K becomes much smaller, i.e. below 85.

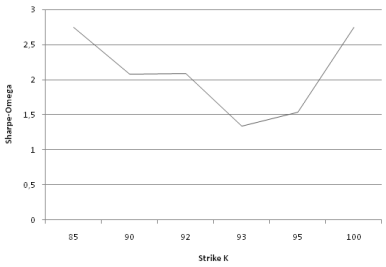


Fig.7. Sharpe-Omega for K in $[80, 100]$

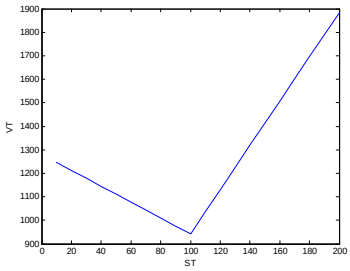


Fig.8. Optimal portfolio profile

Optimal allocation for $K = 100$ shows 94% invested in the risky instrument and 6% as a hedge on the downside. Portfolio expected return M is higher than in the previous case with a lower minimum at 6%.

3.3 Maximizing the Sharpe Omega Ratio with an equity, one put and one call

In this section, we add a call option to the previous portfolio (excluding risk-free instrument). This second structured financial product is a portfolio which involves β risky assets combined with α calls and γ puts written on the risky asset. The time period is $[0, T]$. Thus, the portfolio value V_T is given at maturity T by:

$$V_T = \alpha(S_T - K_C)^+ + \beta S_T + \gamma(K_P - S_T)^+, \tag{29}$$

where K_C is the strike price of Call and K_P is the strike price of the Put.

The value of the initial investment amount is given by:

$$V_0 = \alpha C_0(K_c) + \beta S_0 + \gamma P_0(K_P). \tag{30}$$

3.3.1 Conditions on weights

We determine the optimal combination of call/put and stock with respect to the Sharpe Omega measure. We search for the optimal amounts α^*, β^* and γ^*. We solve this problem under the budget constraint V_0:

$$V_0 = \alpha C_0(K_c) + \beta S_0 + \gamma P_0(K_P). \tag{31}$$

In this case, we find (β, γ) such that:

$$\begin{cases} V_0 - \alpha C_0(K_c) = \beta S_0 + \gamma P_0(K_P), \\ M - \alpha \mathbb{E}_{\mathbb{P}}\left[S_T - K_C\right]^+ = \beta S_0 e^{\mu T} + \gamma \mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+ \end{cases}$$

Thus, we deduce the value β^*:

$$\beta^* = \frac{\begin{vmatrix} M - \alpha \mathbb{E}_{\mathbb{P}}\left[S_T - K_C\right]^+ & \mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+ \\ V_0 - \alpha C_0(K_c) & P_0(K_P) \end{vmatrix}}{\Delta}, \tag{32}$$

where

$$\Delta = \begin{vmatrix} S_0 e^{\mu T} & \mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+ \\ S_0 & P_0(K_P) \end{vmatrix}$$

We obtain:

$$\beta = \alpha \left(\frac{C_0(K_c)\mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+ - P_0(K_P)\mathbb{E}_{\mathbb{P}}\left[S_T - K_C\right]^+}{\Delta} \right) \tag{33}$$

$$+ \frac{MP_0\left(K\right) - V_0 \mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+}{\Delta} \tag{34}$$

Therefore, β^* can be written as $a_{12}\alpha + b_{12}$, with

$$a_{12} = \frac{C_0(K_c)\mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+ - P_0(K_P)\mathbb{E}_{\mathbb{P}}\left[S_T - K_C\right]^+}{\Delta} \tag{35}$$

and

$$b_{12} = \frac{MP_0\left(K\right) - V_0 \mathbb{E}_{\mathbb{P}}\left[K_P - S_T\right]^+}{\Delta} \tag{36}$$

Similarly, we determine γ^*:

$$\gamma^* = \frac{\begin{vmatrix} S_0 e^{rT} & M - \alpha \mathbb{E}_{\mathbb{P}}\left[S_T - K_C\right]^+ \\ S_0 & V_0 - \alpha C_0(K_c) \end{vmatrix}}{\Delta}, \tag{37}$$

and γ^* can be written as $a_{22}\alpha + b_{22}$, with

$$a_{22} = \frac{S_0 \mathbb{E}_{\mathbb{P}}\left[S_T - K_C\right]^+ - C_0(K_c) S_0 e^{\mu T}}{\Delta}, \tag{38}$$

and

$$b_{22} = \frac{V_0 S_0 e^{\mu T} - S_0 M}{\Delta}. \tag{39}$$

3.3.2 Constraints on portfolio optimization

We keep the positivity constraint on V_T, as in the first structured portfolio. This yields to the following conditions on α:

Positivity for $S_T = 0$:

1) $\gamma \geq 0$, which is equivalent to $a_{22}\alpha + b_{22} \geq 0$.

Positivity for $S_T = K_P$:

$$2)\ \alpha \left(K_P - K_C\right)^+ + \beta K_P \geq 0, \tag{40}$$

which is equivalent to

$$\left(\left(K_P - K_C\right)^+ + a_{12} K_P\right) \alpha + K_P b_{12} \geq 0.$$

Positivity for $S_T = K_C$:

$$3)\ \beta K_C + \gamma \left(K_P - K_C\right)^+ \geq 0$$

which is equivalent to

$$\left(a_{22} \left(K_P - K_C\right)^+ + a_{12} K_C\right) \alpha + K_C b_{12} + b_{22} \left(K_P - K_C\right)^+ \geq 0.$$

and finally:

Positivity for $S_T > Max(K_C, K_P)$:

$$4)\ \beta + \alpha \geq 0, \text{ which is equivalent to } \left(a_{12} + 1\right) \alpha + b_{12} \geq 0. \tag{41}$$

In what follows, we deeply analyze the case $K_P = 100$ (at-the-money put) and $K_C = 115$ (out-of-the money call). Figure 9 provides the efficient frontier (Put, Expected return). When we maximize the Sharpe-Omega ratio at given expected return M, the put price exhibits the following variations: it is decreasing when M varies between 1000 and 1035.5 and increasing for M greater than 1035.5. Figure 10 provides the optimal weights according to expected value M.

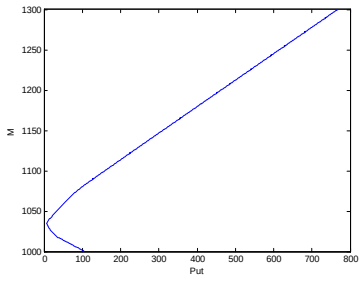


Fig.9. Efficient frontier

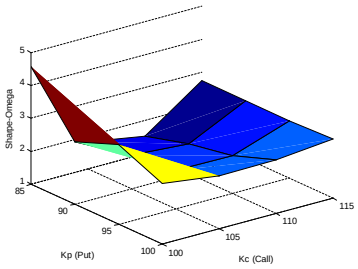
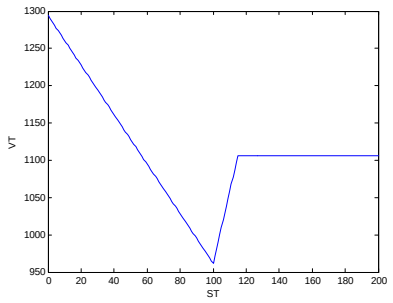
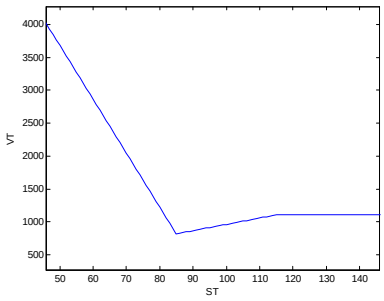


Fig.10. Optimal allocations w.r.t. M

Maximum Sharpe-Omega ratio is reached for $M^* = 1035.5$. Portfolio payoff remains convex with higher slope on its right side. Thus, the portfolio expected return is 3.5% with a limited downside from its initial value at 3.8%. The optimal allocation is: -2.05% on call option, 96.2% on risky instrument and 5.86% on put option.



$K_C = 115, K_P = 100$



$K_C = 100, K_P = 85$

Fig.11.Optimal portfolio payoff
for call-put portfolio

As shown in Figure 12, optimal Sharpe-Omega ratio variations are very similar to those in the first structured portfolio case. Moreover, the Sharpe-Omega ratio calculated for $K_P = 100$ is not sensitive to K_C. Looking more closely to optimal portfolio weightings, call option allocation varies between -7% (when K_P is deeply out-of-the money) to 20%. When call option weight is positive, it provides additional "beta" to the portfolio for high values of the risky instrument, in exchange of lower portfolio minimum. In the opposite case, it neutralizes the portfolio "beta".

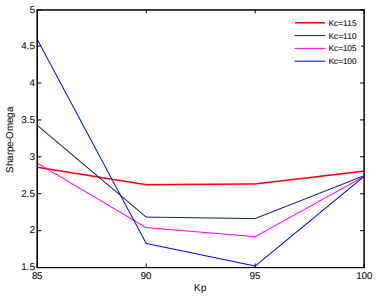


Fig.12. Sharpe-Omega w.r.t. K_P

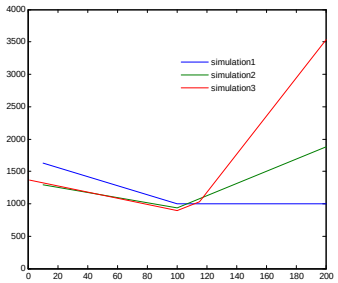


Fig.13. The three profiles

To summarize, we plot in Figure 13 the three optimal porfolio payoffs:
Simulation 1: portfolio with money market account ($K = 100$),
Simultation 2: portfolio without money market account ($K = 100$),
Simulation 3: portfolio call/put ($Kp = 100, Kc = 100$).

4 Conclusion

In this paper, we have examined performance maximization of plain-vanilla structured products. For this purpose, we have considered the Omega or Sharpe Omega performance ratios introduced by Keating and Shadwick (2002) and by Kazemi et al. (2004). Optimal payoffs are no longer increasing and concave as for the Sharpe ratio maximization case illustrated by Goetzmann et al. (2002). They can be always decreasing (it means that the investor wants a high payoff only when the financial market drops significantly) or decreasing then increasing such as straddles. This latter kind of strategy is based on anticipations of relatively extreme events (significant drops or rises of the risky asset due to high volatility). Investors bear losses for relatively stable financial markets. This is in line with previous results on portfolio optimization within rank dependent utility, as quoted by Prigent (2008). Further extensions can take account of more complex derivatives, such as exotic options and also dynamic portfolio strategies.

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5 Appendix

Appendix 1: Computation of the risk component $E_{\mathbb{P}}\left[(L-V_T)^+\right]$

Case (a.2).
We have:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L-V_T)^+\right] &= \int_0^K (L-\alpha B_T-\beta s-\gamma K+\gamma s)\, d\mathbb{P}_{S_T}(s) \\ &\quad + \int_K^{k_2} (L-\alpha B_T-\beta s)\, d\mathbb{P}_{S_T}(s) \\ &= \int_0^K (L-\alpha B_T-\gamma K)\, d\mathbb{P}_{S_T}(s) + \int_0^K (\gamma-\beta)\, s d\mathbb{P}_{S_T}(s) \\ &\quad + \int_K^{k_2} (L-\alpha B_T)\, d\mathbb{P}_{S_T}(s) + \int_K^{k_2} (-\beta s)\, d\mathbb{P}_{S_T}(s)\end{aligned}$$

Moreover, we have:

$$1) \int_0^K (L-\alpha B_T-\gamma K)\, d\mathbb{P}_{S_T}(s) = (L-\alpha B_T-\gamma K)\, \phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right),$$

$$2) \int_0^K (\gamma-\beta)\, s d\mathbb{P}_{S_T}(s) = (\gamma-\beta)\exp\left(\overline{z}+\frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z}-\ln K+\sigma^2 T}{\sigma\sqrt{T}}\right),$$

$$3) \int_K^{k_2} (L-\alpha B_T)\, d\mathbb{P}_{S_T}(s) = (L-\alpha B_T)\left[\phi\left(-\frac{\overline{z}-\ln k_2}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right)\right],$$

$$4) \int_K^{k_2} (-\beta s)\, d\mathbb{P}_{S_T}(s) = -\beta\exp\left(\overline{z}+\frac{\sigma^2}{2}T\right) W_{a_2},$$

where W_{a_2} is equal to:

$$\left[\phi\left(-\frac{\overline{z}-\ln k_2+\sigma^2 T}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln K+\sigma^2 T}{\sigma\sqrt{T}}\right)\right].$$

Finally, we deduce:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L-V_T)^+\right] &= -\gamma K\phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right)+(L-\alpha B_T)\,\phi\left(-\frac{\overline{z}-\ln k_2}{\sigma\sqrt{T}}\right)\\ &\quad+\gamma\exp\left(\overline{z}+\frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z}-\ln K+\sigma^2T}{\sigma\sqrt{T}}\right)\\ &\quad-\beta\exp\left(\overline{z}+\frac{\sigma^2}{2}T\right)\phi\left(-\frac{\overline{z}-\ln k_2+\sigma^2T}{\sigma\sqrt{T}}\right)\end{aligned}$$

Case (b.1).

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L-V_T)^+\right] &= \int_{k_1}^{K}(L-\alpha B_T+(\gamma-\beta)\,s-\gamma K)\,d\mathbb{P}_{S_T}(s)\\ &\quad+\int_{K}^{k_2}(L-\alpha B_T-\beta s)\,d\mathbb{P}_{S_T}(s)\end{aligned}$$

The first term can be written as follows:

$$\begin{aligned}i_1 &= (L-\alpha B_T-\gamma K)\int_{k_1}^{K}d\mathbb{P}_{S_T}(s)+(\gamma-\beta)\int_{k_1}^{K}s d\mathbb{P}_{S_T}(s),\\ &= (L-\alpha B_T-\gamma K)\left[\phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln k_2}{\sigma\sqrt{T}}\right)\right]\\ &\quad+(\gamma-\beta)\exp\left(\overline{z}+\frac{\sigma^2T}{2}\right)\left[\phi\left(-\frac{\overline{z}-\ln K+\sigma^2T}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln k_1+\sigma^2T}{\sigma\sqrt{T}}\right)\right].\end{aligned}$$

The second term is equal to:

$$\begin{aligned}i_2 &= (L-\alpha B_T)\int_{K}^{k_2}d\mathbb{P}_{S_T}(s)-\beta\int_{K}^{k_2}s d\mathbb{P}_{S_T}(s)\\ &= (L-\alpha B_T)\left[\phi\left(-\frac{\overline{z}-\ln k_2}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right)\right]\\ &\quad-\beta\exp\left(\overline{z}+\frac{\sigma^2T}{2}\right)\left[\phi\left(-\frac{\overline{z}-\ln k_2+\sigma^2T}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln K+\sigma^2T}{\sigma\sqrt{T}}\right)\right]\end{aligned}$$

Case (b.2).

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(L-V_T)^+\right] &= \int_{0}^{k_2}(L-\alpha B_T-(\beta-\gamma)\,\mathbb{P}_{S_T}(s)-\gamma K)\,d\mathbb{P}_{S_T}(s)\\ &\quad+\int_{K}^{k_2}(L-\alpha B_T-\beta s)\,d\mathbb{P}_{S_T}(s)\end{aligned}$$

The first term is expressed as follows:

$$\begin{aligned}&= (L-\alpha B_T-\gamma K)\int_{0}^{K}d\mathbb{P}_{S_T}(s)-(\beta-\gamma)\int_{0}^{K}s d\mathbb{P}_{S_T}(s)\\ &\quad(L-\alpha B_T-\gamma K)\,\phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right)-(\beta-\gamma)\,\phi\left(-\frac{\overline{z}-\ln K+\sigma^2T}{\sigma\sqrt{T}}\right)\end{aligned}$$

The second term is expressed as follows:

$$\begin{aligned}
&= (L-\alpha B_T)\int_K^{k_2} d\mathbb{P}_{S_T}(s) - \beta\int_K^{k_2} s d\mathbb{P}_{S_T}(s) \\
&= (L-\alpha B_T)\left[\phi\left(-\frac{\overline{z}-\ln k_2}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln K}{\sigma\sqrt{T}}\right)\right] \\
&\quad -\beta\exp\left(\overline{z}+\frac{\sigma^2 T}{2}\right)\left[\phi\left(-\frac{\overline{z}-\ln k_2+\sigma^2 T}{\sigma\sqrt{T}}\right)-\phi\left(-\frac{\overline{z}-\ln K+\sigma^2 T}{\sigma\sqrt{T}}\right)\right]
\end{aligned}$$

Appendix 2: calculation of amounts β and γ

From equations (20), we deduce:

$$\beta = \frac{\begin{vmatrix} V_0-\alpha B_0 & P_0(K) \\ M-\alpha B_0 e^{rT} & BS(\mu)e^{\mu T} \end{vmatrix}}{\Delta},$$

where

$$\begin{aligned}
\Delta &= \begin{vmatrix} S_0 & P_0(K) \\ S_0 e^{\mu T} & BS(\mu)e^{\mu T} \end{vmatrix} \\
&= S_0\left(BS(\mu)-P_0(K)e^{\mu T}\right) \qquad (42)
\end{aligned}$$

Then, we have:

$$\beta = \alpha\left(\frac{-BS(\mu)e^{\mu T}B_0+P_0(K)B_0e^{rT}}{\Delta}\right)+\frac{BS(\mu)V_0e^{\mu T}-P_0(K)M}{\Delta}.$$

Similarly we determine γ:

$$\gamma = \frac{\begin{vmatrix} S_0 & V_0-\alpha B_0 \\ S_0 e^{\mu T} & M-\alpha B_0 e^{rT} \end{vmatrix}}{\Delta},$$

from which, we deduce:

$$\gamma = \alpha\left(\frac{-S_0B_0e^{rT}+B_0S_0e^{\mu T}}{\Delta}\right)+\frac{S_0M-S_0e^{\mu T}V_0}{\Delta}.$$

Appendix 3 : Signs of $\mathbf{a_{11}}$, $\mathbf{a_{21}}$, $b_1(M)$ and $b_2(M)$

Parameters a_{11} and a_{21} are non positive.
We have:

$$\begin{aligned}
a_{11} &= \frac{-BS(\mu)e^{\mu T}B_0+P_0(K)B_0e^{rT}}{\Delta}, \\
&= \frac{B_0}{\Delta}\left[P_0(K)e^{rT}-BS(\mu)e^{\mu T}\right].
\end{aligned}$$

The sign of Δ depends on $[BS(\mu) - P_0(K)]$. However, to look for the sign of a_{11}, we first determine the sign of $\frac{\partial P(r)}{\partial r}$. We have:

$$\begin{aligned} \frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} - TKe^{-rT}, \\ &= Ke^{-rT}[N(d_2) - 1] \end{aligned}$$

Since $[N(d_2) - 1] < 0$, we deduce that $TP(r) + \frac{\partial P}{\partial r} = T\left(P(r) + Ke^{-rT}[N(d_2) - 1]\right)$.

Additionally,

$$P(r) = S_0 N(d_1) - Ke^{-rT}[N(d_2) - 1] - S_0 + Ke^{-rT}$$

$\Rightarrow$

$$P(r) + Ke^{-rT}[N(d_2) - 1] = S_0[N(d_1) - 1]$$

$[N(d_1) - 1] < 0$. Thus, $e^{-rT}\ P(r)$ is decreasing.

If $\mu > r$ then $BS(\mu) < P_0(K)$. Hence, the sign of a_1 is negative since $P_0(K)e^{rT} > BS(\mu)e^{\mu T}$ and $\Delta < 0$.

Similarly, we determine the sign of a_2.

$$a_{21} = \frac{S_0 B_0}{\Delta}\left(e^{\mu T} - e^{rT}\right),$$

If $\mu > r \Rightarrow e^{\mu T} > e^{rT}$, and $\Delta < 0$. Therefore, $a_2 < 0$.

The function $b_{11}(M)$ is always increasing and $b_{21}(M)$ is decreasing.

$$b_{11} = \frac{BS(\mu)V_0 e^{\mu T} - P_0(K)M}{\Delta}$$

If $\mu > r$ and $M > V_0 e^{\mu T}$ (100% in S_T) then $BS(\mu) < P_0(K)$ and $b_1 > 0$.