Stochastic Dominance of Any Type and Any Degree, and Expected Utility: A Unifying Approach

André Lapidus

Stochastic Dominance of Any Type and Any Degree, and Expected Utility: A Unifying Approach

André Lapidus*

2015
(First version: October 2013)

Abstract

This paper provides a general framework for a unifying treatment of stochastic dominance of any degree and of any type (direct or inverse for each final or intermediary level). It gives the conditions for the congruence between stochastic dominance and classes of utility functions in this general framework and shows, as particular cases, the properties of some varieties of stochastic dominance usually neglected in standard literature.

Keywords: stochastic dominance; expected utility; risk; decision.

JEL classification: D81.

1 Introduction

Stochastic dominance as a way for partially ordering probability distributions was introduced in economics and decision theory through some pioneering articles published from the 1960's onwards[1] (Quirk and Saposnik 1962, Hadar and Russell 1969, Hanoch and Levy 1969, Rothschild and Stiglitz 1970, Whitmore 1970) and spawned later an important body of literature. From the very beginning, emphasis was laid on the equivalence between the attitude toward risk of a decision maker expressed by various types of stochastic dominance, and corresponding classes of utility functions, within an expected utility framework. Called "congruence" by Fishburn (1976), this equivalence between a partial order, generated by stochastic dominance, and the intersection between complete orders, generated by a class of utility functions, led to the powerful conclusion that preference for a stochastically dominating distribution of such type might be represented by any utility function belonging to a specific class; and, conversely, that a decision maker endowed with any utility function from this class always prefers stochastically dominating distributions of this type. However, no general formulation was given till now.

The first results (Quirk and Saposnik 1962, Hadar and Russell 1969, Hanoch and Levy 1969, Rothschild and Stiglitz 1970) related first degree stochastic dominance (FSD) to the class of non-decreasing utility functions – that is, to functions whose first derivative was non-negative – so that

*PHARE, University Paris 1 Panthéon-Sorbonne and CNRS – 106-112, boulevard de l'Hôpital – 75647 Paris Cedex 13 – France. E-mail: andre.lapidus@univ-paris1.fr

[1]The first demonstrations on stochastic dominance appeared long before its introduction in economics. Le Breton (1987) pointed out an alternative demonstration concerning second degree stochastic dominance when both distributions have the same mean, published by Karamata as early as 1932.

both risk-averse and risk-seeking decision makers preferred first degree stochastically dominating distributions[2]. In the last three papers (Hadar and Russell 1969, Hanoch and Levy 1969, Rothschild and Stiglitz 1970), it was also shown that second degree stochastic dominance (SSD) was linked to the class of non-decreasing concave utility functions – whose first and second derivatives are respectively positive and negative – which characterize risk-aversion for expected utility decision makers. This result was extended to a more specific type of risk-aversion, taking skewness into account, by Whitmore (1970), who linked third degree stochastic dominance (TSD) to the class of non-decreasing concave utility functions with positive third derivative, provided the mean of the dominating distribution was at least equal to that of the dominated one. Later on, such attitude toward risk has been named "prudence" (Kimball 1990), and unified accounts of the three first degrees of stochastic dominance could be provided (see, for instance, Thorlund-Petersen 2001). Again, the result on third degree stochastic dominance was generalized to n-th degree stochastic dominance, which was related to utility functions whose odd and even derivatives are respectively positive and negative, till the n-th degree (Fishburn 1976), therefore allowing a broader exploration of these refinements of the attitude toward risk involved similarly in higher degree stochastic dominance and in the signing of higher degree derivatives of the utility function, like what was called "temperance" (4th degree, after Kimball 1992) or "edginess" (5th degree, Lajeri-Chaherli 2004).

Symmetrically, characterizations of risk-seeking appeared some years later, giving rise to what was to be called (in spite of an imperfectly fixed vocabulary) "inverse" stochastic dominance[3]. Goovaerts, De Vylder and Haezendonck (1984) introduced second degree inverse stochastic dominance (SISD), which amounts to what Levy (2006, pp. 126-130) called "risk-seeking stochastic dominance" (RSSD), as related to the class of non-decreasing convex utility functions (with non-negative first and second derivatives). Like in the case of direct dominance, taking skewness into account led to third degree inverse stochastic dominance, which was divided by Zaras (1989) between two types: third degree type 1 inverse stochastic dominance (TISD1), linked to the class of convex utility functions with non-positive third derivative (Zaras 1989) therefore denoting imprudence; and third degree type 2 inverse stochastic dominance (TISD2), which also corresponds to convex utility functions, but with non-negative third derivative (Goovaerts, De Vylder and Haezendonck 1984) which might be interpreted as prudence, like for TSD[4].

Since the pioneering works of the 1960's and 70's, the concept of stochastic dominance has been subjected to several refinements and extensions: quantile approach, introduction of riskless assets, extension to the measure of inequalities, multicriteria decision, almost dominance or fuzzy measures, consistency with non-expected utility theories like rank-dependent utility or cumulative prospect theory, etc... A review of this literature can be found in Guo (2012), and in the books by Sriboonchitta, Wong, Dhompongsa and Nguyen (2010) and by Levy (2006). But what seems to have been the most salient evolution concerns the way the orders generated by stochastic dominance

[2]An intuitive knowledge of first degree stochastic dominance in relation to decision seems to have been widely spread long before the contributions of 1969 and 1970. As Pradier (2006) pointed out, Jacob Bernoulli, for instance, argued as early as the very beginning of 18th century, in the fourth part of his *Ars Conjectandi*, that "what may be advantageous in one case, and can never harm, should be preferred [*praeferendum est*] to what is in no case beneficial or harmful", and related this to popular wisdom expressed in a German saying, "*Hilffdt es nicht / so schadt es nicht*" (Bernoulli 1713, p. 320).

[3]For instance, in a pioneering paper where they applied stochastic dominance to the question of of inequality measures, Muliere and Scarsini (1989) named "inverse" stochastic dominance the integration of differences between the inverses of cumulative distribution functions, F^{-1} and G^{-1}, and not between decumulative functions.

[4]For a systematic account of risk attitudes involved in the signing of the various derivatives of the utility function, see Eeckhout and Schlesinger 2006.

were considered. The original approach leads to view stochastic dominance of various types as generating possible partial preference preorders over a set of lotteries. This justifies raising the question of the congruence with a class of utility functions within a context of expected utility. However, once the idea of congruence has been accepted in principle, attention seems to have shifted toward the nature of the concept of stochastic dominance which allows the efficient selection of a lottery among a subset of lotteries, according to some utility functions belonging to the congruent class. Typical of this appraoach is Post and Kopa (2013) paper, which allows for comparing a given lottery with a discrete set of alternative lotteries, or with a set of linear combinations of these lotteries, on the basis of a generalization of the concept of convex stochastic dominance, which Fishburn introduced as early as 1974.

This paper goes a step back to the original approach, acknowledging that the question of congruence has been solved only for the few cases noted above. It follows on from Fishburn's solution, which concerned n-th degree direct stochastic dominance (Fishburn 1976) but gives up this limitation to a specific type of stochastic dominance. Therefore, it first provides a unifying treatment, now permitting any type of stochastic dominance, direct or inverse, and of any degree, in relation to classes of utility functions defined by the signing of their successive derivatives (section 2). This allows giving general conditions of congruence, that is conditions for the representation of each kind of stochastic dominance by a class of utility functions and reciprocally (section 3).

2 A formal framework

Denote X and Y two distinct random variables with support $[a,b] \subset \mathbb{R}$. Assume that their respective distribution functions, F and G, are absolutely continuous, so that they can be represented by their density of probability functions f and g[5]. Stochastic dominance of degree i ($i = 1, ..., n$) of f over g (X over Y) amounts to the non-positivity of an index of dominance for each value of x on $[a,b]$. This index, which is said *final* when $i = n$ and *intermediary* otherwise, is the i-th integral of $f(x) - g(x)$, each step of integration being either from a to x – *direct* dominance – or from x to b – *inverse* dominance. The situation usually favored in the literature, where each step of integration from 1 to n is direct, will be termed hereafter "complete direct dominance" of degree n. Symmetrically, in an expected utility framework, preference given to f over g amounts to the non-negativity of the difference between expected utilities, $E_u(f) - E_u(g)$, which depends on the properties of a utility function u – assumed n-th differentiable – expressed in the signs of its successive derivatives, $u_1(x), ...u_n(x)$.

The formal framework within which the relation between stochastic dominance and expected utility is investigated results from the definitions of (1) the index and condition of stochastic dominance; (2) the classes of utility functions and of resulting preferences according to expected utility; and (3) a transformation procedure from the degrees of integration of the index of stochastic dominance to the degrees of derivation of the corresponding utility function.

Definition 1. Stochastic dominance

Let $A \subseteq [n]$, where $[n]$ is the set of integers $\{1, ..., n\}$. Then:

[5] Usual regularities, like absolute continuity on a compact interval of $\mathbb{R}$, are assumed throughout this paper for the sake of simplicity, in order to escape possible complications (requiring the use of Riemann-Stieltjes or Lebesgue-Stieltjes integrals) when Lebesgue's criterion for Riemann integrability does not hold for f and g. See the clarification and restatement of the first results of Hanoch and Levy (1969) and Hadar and Russell (1971) by Tesfatsion (1976).

(i) The stochastic dominance index *of degree $i \in [n]$, denoted $H^i_{A,n}(x)$ is defined by*[6]*:*

$$H^i_{A,n}(x) = \mathbb{1}_A(i) \int_a^x H^{i-1}_{A,n}(y)\,\mathrm{d}y + (1 - \mathbb{1}_A(i)) \int_x^b H^{i-1}_{A,n}(y)\,\mathrm{d}y$$
$$\left(\text{with } H^0_{A,n}(x) = f(x) - g(x)\right).$$

This index is final *if $i = n$ and* intermediary *if $i < n$.*

(ii) Let $fD_{A,n}g$ denote the stochastic dominance of f over g at final degree n, direct *for all degrees in A and* inverse *for all degrees not in A. The condition of stochastic dominance of f over g is defined as:*

$$fD_{A,n}g \Leftrightarrow \forall x \in [a,b],\ H^n_{A,n}(x) \leq 0$$

(since the two random variables, X and Y, are distinct, $f \neq g$, and strict inequality holds for at least one value of x).

Note. Throughout this paper, subscripts n or A in $H^i_{A,n}$ are omitted when the context avoids any misunderstanding.

Example 1. Assume $[n] = \{1, ...l, ...m, ...n\}$ with $1, l, n \in A$ and $m \in \complement_{[n]}A$ (that is, to the complement of A with respect to $[n]$). The dominance index of final degree n and of intermediate degrees $1, l, m$ results from repeated integrations of $f(x) - g(x)$, where the first integration is from a to x_1, the l-th from a to x_l, the m-th from x_m to b, and the n-th from a to x:

$$H^n_{\{1,...l,...n\}}(x) = \int_a^x ... \left[\int_{x_m}^b ... \left[\int_a^{x_l} ... \left[\int_a^{x_1} (f(y) - g(y))\,\mathrm{d}y\right] ...\mathrm{d}x_{l-1}\right] ...\mathrm{d}x_{m-1}\right] ...\mathrm{d}x_{n-1}.$$

If $H^n(x) \leq 0$ for each value of x on $[a,b]$, f is said to dominate stochastically g, directly at final degree n and at intermediary degrees 1 and l, and inversely at intermediary degree m.

Remark 1. Let $A_{(i)}$ and $A_{\backslash(i)}$ be identical subsets of $[n]$, except that i belongs to $A_{(i)}$ and not to $A_{\backslash(i)}$. By construction, the following properties always hold:

$$H^i_{A_{(i)}}(a) = H^i_{A_{\backslash(i)}}(b) = 0 \quad (a)$$
$$H^i_{A_{(i)}}(b) = H^i_{A_{\backslash(i)}}(a) \quad (b)$$
$$H^i_{A_{(i)}}(x) = H^i_{A_{\backslash(i)}}(a) - H^i_{A_{\backslash(i)}}(x) \quad (c)$$
$$H^i_{A_{\backslash(i)}}(x) = H^i_{A_{(i)}}(b) - H^i_{A_{(i)}}(x) \quad (d)$$
$$H^i_{A_{(1)}}(x) = -H^i_{A_{\backslash(1)}}(x) \quad (e)$$

Remark 2. Standard literature often favors situations of *complete* direct stochastic dominance, where the complementary of A, $\complement_{[n]}A$, is empty. For instance: first (FSD), second (SSD), third (TSD) and n-th (NSD) degree stochastic dominance respectively correspond to configurations where the bipartitions of $[n]$ are respectively such that $\left(A, \complement_{[1]}A\right) = (\{1\}, \emptyset)$, $\left(A, \complement_{[2]}A\right) = (\{1,2\}, \emptyset)$, $\left(A, \complement_{[3]}A\right) = (\{1,2,3\}, \emptyset)$ and $\left(A, \complement_{[n]}A\right) = (\{1,...n\}, \emptyset)$. The possibility that $\complement_{[n]}A$ be not empty (that is, that f stochastically dominates g *inversely* at some intermediary or final degree) is typically dealt with in order to explore risk-seeking, through either second degree inverse stochastic

[6] Recall that the indicator function $\mathbb{1}_A$ in Definition 1 is an application from $[n]$ to $\{0,1\}$ which, for $i \in [n]$, yields 1 if i belongs to A, and 0 if it does not.

dominance (SISD or RSSD - risk-seeking stochastic dominance), where $\left(A, \complement_{[2]}A\right) = (\{1\}, \{2\})$, or third degree inverse stochastic dominance, either of type 1 (TSID1) when the third integration is from a to x so that $\left(A, \complement_{[3]}A\right) = (\{1,3\}, \{2\})$, or of type 2 (TISD2) when the third integration is from x to b so that $\left(A, \complement_{[3]}A\right) = (\{1\}, \{2,3\})$.

The construction of relevant classes of n-differentiable utility functions u defined on $[a,b]$ obeys the same principles as those which have governed the construction of an index of stochastic dominance. The starting point is the identification of the sets of degrees of derivation of a utility function, corresponding to either positive or negative derivatives.

Definition 2. Utility

Let $B \subseteq [n]$. Then:

(i) Define a set $U_{B,n}$ of n-differentiable utility functions as:

$$U_{B,n} = \{u : [a,b] \to \mathbb{R} \text{ such that } \forall i \in B, \forall x \in [a,b], u_i(x) \geq 0 \text{ and } \forall i \in \complement_{[n]}B, \forall x \in [a,b], u_i(x) \leq 0\}$$

(with for all i, strict inequalities for at least one x).

(ii) Following the expected utility approach, all decision makers whose utility function belongs to $U_{B,n}$ are said to prefer f to g when for each of them, the expected value of the utility of f is not smaller than that of g. Their common preference is denoted $fR_{B,n}g$:

$$fR_{B,n}g \Leftrightarrow \forall u \in U_{B,n}, E_u(f) - E_u(g) \geq 0.$$

Remark 3. The signing of the derivatives of the utility function is usually linked to some typical attitudes toward risk through the ranking of random variables according to their expected utility. For instance, it is well-known that increasing utility functions ($1 \in B$) characterize decision-makers with monotonous increasing preferences, whatever their attitude toward risk. When $2 \in \complement_{[n]}B$, the decision-maker is risk-averse and, conversely, he or she is risk-seeking when $2 \in B$. The intuitive meaning of $3 \in B$ is this of "prudence" (Kimball 1990), so that $3 \in \complement_{[n]}B$ corresponds to "imprudence". Higher degrees of derivation are not that easy to interpret. However, $4 \in \complement_{[n]}B$ and $4 \in B$ are currently viewed as, respectively, "temperance" (Kimball 1992) and "intemperance". Similarly, $5 \in B$ and $5 \in \complement_{[n]}B$ correspond to "edginess" (Lajeri-Chaherli 2004) and to what might be called "calmness". Typical utility functions like the logarithmic ones, which give rise to monotone increasing preferences, risk-aversion, prudence, temperance, edginess and so on, are characterized by alternatively positive and negative derivatives, and were sometimes called "mixed risk averse" utility functions (Caballé and Pomansky, 1996): $B = \{1,3,...\}$ and $\complement_{[n]}B = \{2,4,...\}$). Eeckhoudt and Schlessinger (2006) have shown the equivalence between such higher order risk attitudes and preferences over particular classes of lottery pairs, involving zero-mean independent noise random variables.

The relation between stochastic dominance and expected utility is taken up through a relation between two elements A and B of the powerset of $[n]$. Consider the Procedure AB hereafter, which allocates i ($i = 1,...,n$) between B and $\complement_{[n]}B$ as a degree of derivation, according to its belonging, as a degree of integration, to A or $\complement_{[n]}A$, and to the belonging of $i-1$ to B or $\complement_{[n]}B$:

Procedure AB

- *For all* $i \in A$
$$\begin{cases} \text{if } i-1 \in B, \text{ then } i \in \complement_{[n]}B \\ \text{if } i-1 \in \complement_{[n]}B \text{ or } i=1, \text{ then } i \in B \end{cases}$$
- *For all* $i \in \complement_{[n]}A$
$$\begin{cases} \text{if } i-1 \in \complement_{[n]}B \text{ or } i=1, \text{ then } i \in \complement_{[n]}B \\ \text{if } i-1 \in B, \text{ then } i \in B \end{cases}$$

Definition 3. Transformation
The transformation of $i = 1, ..., n$, as degrees of integration belonging to either A or $\complement_{[n]}A$, into degrees of derivation belonging to either B or $\complement_{[n]}B$, is performed by

$$\phi : 2^{[n]} \to 2^{[n]},\ A \mapsto \phi(A) = B \tag{1}$$

3 Congruence between stochastic dominance and expected utility

The issue is now that of the conditions of consistency between two partial orders: according to stochastic dominance, $D_{A,n}$, and to expected utility, $R_{B,n}$. Though presented differently, such consistency was called “congruence” by P. Fishburn (1976, p. 303):

Definition 4. Congruence
A stochastic dominance order $D_{A,n}$ and an expected utility order $R_{B,n}$ are congruent *when, for each f and g,*

$$fD_{A,n}g \Leftrightarrow fR_{B,n}g.$$

It will be shown (Proposition 1) that, provided additional conditions on the bounds of the distributions are satisfied, congruence is achieved when B is the image of A through the transformation ϕ defined in (1). The structure of the relation between the orders generated by stochastic dominance and expected utility is further investigated in Proposition 2.

The following lemma shows that each possible allocation of the degrees of integration corresponds through ϕ to an allocation of the degrees of derivation, and reciprocally:

Lemma 1. $\forall A \in 2^{[n]}, \exists B \in 2^{[n]} : B = \phi(A)$
and $\forall B \in 2^{[n]}, \exists A \in 2^{[n]} : B = \phi(A)$.

Proof. Check with Procedure AB that ϕ in (1) is a bijection of $2^{[n]}$ into itself. □

A general condition for congruence between stochastic dominance and expected utility orders is presented in the following proposition.

Proposition 1. *For all $A, B \in 2^{[n]}$ such that $B = \phi(A)$, the two following propositions are equivalent for all f and g:*

1. *$fR_{B,n}g$ (expected utility).*
2. *$fD_{A,n}g$ (stochastic dominance)*

and, if $n \geq 3$*, for all* $i = 2, ..., n-1$*:*
$H^i_{A,n}(\gamma_i) \leq 0$ *(where* $\gamma_i = \mathbb{1}_A(i)\, b + (1 - \mathbb{1}_A(i))\, a$*) (conditions on upper or lower bounds).*

Proof. See Appendix. □

An intuitive interpretation of Proposition 1 is that provided conditions on bounds are satisfied, the partial order on random variables generated by any type of stochastic dominance, say $D_{A,n}$, is identical to the intersection between all the complete preference preorders on random variables underlying all the possible utility functions belonging to $U_{B,n}$, when $B = \phi(A)$. Proposition 1 may also be viewed as establishing a link between subjective assessments and objective properties. It means that on the one hand, what all possible decision makers, characterized by a utility function from the same class, have in common evidently rests on their respective subjective preferences; but, on the other hand, this common ranking of random variables may alternatively be viewed as model-free, that is, as depending only on some objective properties of the distribution functions expressed in stochastic dominance.

We know that in the case of complete direct n-th degree stochastic dominance, the conditions on upper bounds can be viewed as algebraic combinations of the differences between the successive moments around zero of the distributions (Jean and Helms 1988). A typical illustration, anticipated by Whitmore (1970), is that in order to have congruence between complete direct third degree stochastic dominance ($A = \{1, 2, 3\}$) and the class of increasing concave utility functions with a non-negative third derivative ($B = \{1, 3\}$), the difference between means (first moments), $E(f) - E(g)$, had to be non-negative[7]. The conclusion of Jean and Helms (1988) still holds in the case of Proposition 1 where dominance is not necessarily direct at each step since, by construction, the value of any index of dominance at the relevant bound is itself an algebraic sum of some complete direct dominance indices of equal and smaller degrees at the upper bound: any index of degree i at the relevant bound is therefore a linear combination of the differences between the moments around zero ${}^jM_f(0) - {}^jM_g(0)$ of all degrees j from 1 to $i-1$.

Example 2. Usual and neglected types of stochastic dominance in relation to expected utility

Such representation of congruence allows finding again the usual formulations of stochastic dominance from the end of the 1960's to the 1980's, and their correspondence with certain classes of utility functions. In return, it also allows finding the neglected or missing categories which can now be put to the fore.

This bringing together is obvious in the case of complete direct dominance in the following usual cases, A_n and B_n denoting elements of the powerset $2^{[n]}$:

FSD (first degree stochastic dominance: Quirk and Saposnik 1962 (discrete case); Hadar and Russell 1969; Hanoch and Levy 1969) corresponds to $A_1 = \{1\}$. It was acknowledged congruent with

$$B_1 = \phi(\{1\}) = \{1\}$$

that is with monotonous non-decreasing preferences, entailing universal preference for the dominating random variable, whatever the decision maker's attitude toward risk.

[7] Recall that $E(f) - E(g) \geq 0$ is already a consequence of complete direct stochastic dominance at degrees 1 or 2.

SSD (second degree stochastic dominance: Hadar and Russell 1969; Hanoch and Levy 1969; Rothschild and Stiglitz 1970 ("mean preserving spread" subcase – MPS)) corresponds to $A_2 = \{1,2\}$, and was acknowledged congruent with

$$B_2 = \phi(\{1,2\}) = \{1\}$$

that is with monotonous non-decreasing preferences and risk-aversion.

TSD (third degree stochastic dominance: Whitmore 1970) corresponds to $A_3 = \{1,2,3\}$, and was acknowledged congruent with

$$B_3 = \phi(\{1,2,3\}) = \{1,3\}$$

that is with monotonous non-decreasing preferences, risk-aversion, positive skewness-seeking and negative skewness-aversion (typically, a decision maker interested in insuring his or her house, and in buying public lotteries tickets). This corresponds to what is usually called "prudence" since Kimball (1990)'s paper, in relation to the positive sign of the third derivative of the utility function.

NSD (n-th degree stochastic dominance: Fishburn 1976) corresponds to complete direct stochastic dominance with $A_n = [n] = \{1,2,3...n\}$ and was acknowledged congruent with

$$B_n = \phi(\{1,2,3,...n\}) = \{1,3,...\}$$

that is, with a class of utility functions whose odd and even derivatives are respectively non-negative and non-positive.

The same bringing together is a bit less immediate in the cases of stochastic dominance of degrees 2 and 3, related to monotonous non-decreasing preferences and risk-seeking, from the point of view of expected utility. These different cases were usually called "inverse" stochastic dominance (see, for example, Zaras 1989), and have in common that $2 \notin A_n$. A technical but simple problem of presentation arises from the fact that the corresponding indices of stochastic dominance generally relied on decumulative distribution functions, of type $\overline{F}(x) = \int_x^b f(y)\,\mathrm{d}y$, instead of cumulative functions $F(x) = \int_a^x f(y)\,\mathrm{d}y$. The resulting successive integrals were therefore computed on the basis of $H^1_{\emptyset,1}(x)$, instead of $H^1_{\{1\},1}(x)$, although the concerned decision makers were endowed with non-decreasing preferences. Nonetheless, since $H^1_{\emptyset,1}(x) = -H^1_{\{1\},1}(x)$, any higher degree of the intermediary or final index is such that, assuming that 1 is an element of A_n, $H^i_{A_n\setminus\{1\},n}(x) = -H^i_{A_n,n}(x)$ (see Remark 1 on Definition 1). So that the non-positivity requirement for the index of stochastic dominance was usually replaced by a non-negativity condition insofar as it was applied to inverse dominance at intermediate or final degree 2. Taking this into account allows establishing the following three correspondences:

SISD (second degree inverse stochastic dominance: Goovaerts, De Vylder and Haezendonck 1984) or **RSSD** (risk-seeking stochastic dominance: Levy 2006) corresponds to $A_2 = \{1\}$, and was acknowledged congruent with

$$B_2 = \phi(\{1\}) = \{1,2\}$$

that is, with monotonous non-decreasing preferences and risk-seeking.

TISD1 (third degree type 1 inverse stochastic dominance: Zaras 1989) corresponds to $A_3 = \{1,3\}$ and was acknowledged congruent with

$$B_3 = \phi(\{1,3\}) = \{1,2\}$$

that is, with monotonous non-decreasing preferences, risk-seeking, positive skewness-aversion and negative skewness-seeking (imprudence).

TISD2 (third degree type 2 inverse stochastic dominance: Goovaerts, De Vylder and Haezendonck 1984) corresponds to $A_3 = \{1\}$ and was acknowledged congruent with

$$B_3 = \phi(\{1\}) = \{1,2,3\}$$

that is, with monotonous non-decreasing preferences, risk-seeking, positive skewness-seeking and negative skewness-aversion (prudence).

However, the three following instances of stochastic dominance are still missing in current literature, and are worth being noted.

1. The first instance is that of situations in which a decision maker would prefer the distribution whose probability to get at least this income is the *smaller* that is, the stochastically dominating random variable at degree 1 when $A_1 = \emptyset$:

FISD (stochastic dominance *inverse* for the degree 1). $A_1 = \emptyset$. This amounts to congruence with

$$B_1 = \phi(\emptyset) = \emptyset$$

that is with non-increasing utility functions. Of course, such situations might seem of little practical interest – except in case the purpose were to consider the behavior toward risk of a decision maker whose objective would be to ruin himself or herself. However, it makes clear that, contrary to a hasty conclusion, though preference for distributions which are stochastically directly dominated at degree 1 by other distributions might seem a bit strange, it is by no way constitutively irrational.

2. But it is also obvious that, by contrast to TSD which has been extensively studied after Whitmore's 1970 paper, in relation to the more familiar idea of DARA (decreasing absolute risk aversion), a second type of third stochastic dominance TSD2, homologous to TISD2, doesn't seem to have aroused special interest:

TSD2 (third degree stochastic dominance, type 2) corresponds to $A_3 = \{1,2\}$. According to Proposition 1, granted that third degree conditions on bounds are satisfied, TSD2 is congruent with

$$B_3 = \phi(\{1,2\}) = \{1\}$$

that is, with monotonous non-decreasing preferences, risk-aversion, positive skewness-aversion and negative skewness-seeking – which amounts to imprudence. After all, we all know people who dislike risk, never buy public lotteries tickets, and nonetheless would reject the idea of insuring their house if they were not legally obliged to subscribe to an insurance contract.

3. Finally, whereas *complete* direct stochastic dominance of any degree n (NSD, where $A_n = \{1, ..., n\}$) has been regularly taken up after Fishburn's 1976 paper (see, for instance, Levy 2006, pp. 131-132), such was not the case for stochastic dominance direct or indirect at any intermediary degree i or final degree n, that is, when A_n is any element of $2^{[n]}$. Filling this gap was the main purpose of this article.

Remark 4. Stochastic dominance orders and classes of utility functions
According to a well-known "sufficient rule" of stochastic dominance orders, if f stochastically dominates g at degree k, it also dominates g at degrees $k+1$, $k+2$, ... (see, for instance, Levy 2006, pp. 119-20, for the relations between complete direct dominances of degrees 1, 2 and 3). Typically, this means that risk-averse and risk-seeking decision makers with positive monotone preferences might disagree on the order between random variables not ordered by first degree direct stochastic dominance, but that they agree on the order of random variables generated by the latter, which is consistent with both direct and inverse stochastic dominance of degree 2. This result can be easily generalized and related to preferences according to expected utility in the following proposition:

Proposition 2. *Let m and n be two integers such that $m \leq n$. If $A_m, B_m \in 2^{[m]}$ and $A_n, B_n \in 2^{[n]}$ satisfy* (a) *and* (b)*:*

(a) $B_m = \phi(A_m)$, *and* $B_n = \phi(A_n)$

(b) $A_m \subseteq A_n, \complement_{[m]} A_m \subseteq \complement_{[n]} A_n$ *or* $B_m \subseteq B_n, \complement_{[m]} B_m \subseteq \complement_{[n]} B_n$

then:

1. $D_{A_m,m} \subseteq D_{A_n,n}$

2. $U_{B_n,n} \subseteq U_{B_m,m}$

(By abuse of notation, $D_{A_m,m}$ and $D_{A_n,n}$ stand for the set-equivalents of the corresponding binary relations).

Proof. First note that, in reason of the definition of ϕ by (1), the two alternative conditions in (b) are equivalent: $A_m \subseteq A_n, \complement_{[m]} A_m \subseteq \complement_{[n]} A_n \Leftrightarrow B_m \subseteq B_n, \complement_{[m]} B_m \subseteq \complement_{[n]} B_n$

1. Obvious, since $H^m(x) \leq 0$ for each x over $[a, b]$ so that its integral, either from a to x, or from x to b, $H^{m+1}(x) \leq 0$, and so on till $H^n(x)$. As a result, $fD_{A_m,m}g \Rightarrow fD_{A_n,n}g$.

2. Observe that, since $B_m \subseteq B_n$ and $\complement_{[m]} B_m \subseteq \complement_{[n]} B_n$, $u \in U_{B_n,n} \Rightarrow u \in U_{B_m,m}$. □

Bringing together the two parts of Proposition 2 shows the structure of the relation between the two partial orders generated either by stochastic dominance, or by expected utility. At the final stage, Proposition 1 makes clear that when conditions on bounds are satisfied, they lead to the same result: $D_{A_m,m}$ and $R_{B_m,m}$ order the same random variables (that is, $D_{A_m,m} = R_{B_m,m}$), just like $D_{A_n,n}$ and $R_{B_n,n}$ (that is, $D_{A_n,n} = R_{B_n,n}$) and it can be concluded from Proposition 2 (1.) that their set is expanding from m to n. However, Proposition 2 (2.) shows that the same conclusion is obtained differently from the expected utility point of view. Moving from m to n first generates a contraction: for instance, such utility function, which was included in $U_{B_m,m}$ does not belong any more to $U_{B_n,n}$, because of the signs of its derivatives from $m+1$ to n. But since the resulting partial order R is the intersection between the complete orders underlying each utility functions belonging to U, R depends on "less complete" orders when moving from m to n. So that it is expanding from $R_{B_m,m}$ to $R_{B_n,n}$, whereas U is contracting from $U_{B_m,m}$ to $U_{B_n,n}$.

4 Concluding remarks

The aim of this paper was to set out a framework for analyzing the relations between subjective preferences between random variables founded on specific classes of utility functions, and the model-free, objective proprieties, of the probability distributions of these random variables expressed in corresponding types of stochastic dominance. This framework first consists in the construction of an index of stochastic dominance, which allows either direct or inverse dominance at any intermediate or final degree. A proposition is demonstrated which establishes the conditions of congruence (in the sense of Fishburn 1976) between the orderings generated by stochastic dominance and by classes of utility functions, extending the results of the pioneering works of Hadar and Russell (1969), Henoch and Levy (1969), Whitmore (1970), Fishburn (1976), Goovaerts, De Vylder and Haezendonck (1984) and Zaras (1989). Comparison with previous contributions also leads to the identification of neglected kinds of stochastic dominance, like a third degree type 2 stochastic dominance (TSD2) and the congruent class of utility functions.

Appendix: Congruence between stochastic dominance and expected utility (Proposition (1))

A Preliminaries concerning the construction of the expression of $E_u(f) - E_u(g)$

[*At least till equation (2), this sub-section can be skipped by the readers familiar with the technique of demonstration already used in most pioneering papers since Hadar and Russell (1969) or Hanoch and Levy (1969)*]

The proof of Proposition 1 follows readily from the repeated integration by parts until degree n of $E_u(f) - E_u(g) = \int_a^b u(x)(f(x) - g(x))\,\mathrm{d}x = \int_a^b u(x) H^0_{A,n}(x)\,\mathrm{d}x$, the non-negativity of $E_u(f) - E_u(g)$ for all $u \in U_{B,n}$ being equivalent to $f R_{B,n} g$. The result is quite classical, except that at each step of integration, it has to be given according to the belonging of i to either A or $\complement_{[n]} A$ - which draws on Remark 1 (a)-(e). Starting from the first degree of integration gives:

- Degree 1:

$$E_u(f) - E_u(g) = \int_a^b u(x) H^0_A(x)\,\mathrm{d}x \begin{cases} 1 \in A: \\ \quad = \left[u(x) H^1_A(x)\right]_a^b - \int_a^b u_1(x) H^1_A(x)\,\mathrm{d}x \\ \quad = -\int_a^b u_1(x) H^1_A(x)\,\mathrm{d}x \\ 1 \in \complement_{[n]} A: \\ \quad = -\left[u(x) H^1_A(x)\right]_a^b + \int_a^b u_1(x) H^1_A(x)\,\mathrm{d}x \\ \quad = \int_a^b u_1(x) H^1_A(x)\,\mathrm{d}x. \end{cases}$$

Note that the sign before the integrals in the right-hand side of the equations is negative when 1 belongs to A, and positive when it does not.

- Degree 2:

$$E_u(f)-E_u(g)\begin{cases} 1\in A: \\ \quad =-\int_a^b u_1(x)\,H_A^1(x)\,\mathrm{d}x \\ \quad =-\left[u_1(x)\,H_{\mathrm{A}}^2\ (x)\right]_a^b+\int_a^b u_2(x)\,H_{\mathrm{A}}^2(x)\,\mathrm{d}x \begin{cases} 2\in A: \\ \quad =-u_1(b)\,H_{\mathrm{A}}^2(b) \\ \qquad +\int_a^b u_2(x)\,H_{\mathrm{A}}^2(x)\,\mathrm{d}x \\ 2\in \complement_{[n]}A: \\ \quad =-u_1(a)\,H_{\mathrm{A}}^2\ (a) \\ \qquad -\int_a^b u_2(x)\,H_{\mathrm{A}}^2\ (x)\,\mathrm{d}x \end{cases} \\ 1\in \complement_{[n]}A: \\ \quad =\int_a^b u_1(x)\,H_A^1(x)\,\mathrm{d}x \\ \quad =\left[u_1(x)\,H_{\mathrm{A}}^2\ (x)\right]_a^b-\int_a^b u_2(x)\,H_{\mathrm{A}}^2(x)\,\mathrm{d}x \begin{cases} 2\in A: \\ \quad =u_1(b)\,H_{\mathrm{A}}^2\ (b) \\ \qquad -\int_a^b u_2(x)\,H_{\mathrm{A}}^2\ (x)\,\mathrm{d}x \\ 2\in \complement_{[n]}A: \\ \quad =u_1(a)\,H_{\mathrm{A}}^2\ (a) \\ \qquad +\int_a^b u_2(x)\,H_{\mathrm{A}}^2\ (x)\,\mathrm{d}x. \end{cases} \end{cases}$$

To sum up, the sign before the integral of $u_2(x)\,H_{\mathrm{A}}^2\ (x)$ in the right-hand side of each equation is positive when 2 is in A and when, at degree 1, the sign before the integral of $u_1(x)\,H_A^1(x)$ in the right-hand side is negative; or when 2 is not A and when, at degree 1, the sign before the integral of $u_1(x)\,H_A^1(x)$ in the right-hand side is positive. Conversely, it is negative when 2 is not A and when, at degree 1, the sign before the integral of $u_1(x)\,H_A^1(x)$ in the right-hand side is negative; or when 2 is in A and when, at degree 1, the sign before the integral of $u_1(x)\,H_A^1(x)$ in the right-hand side is positive. On the other hand, the sign before $u_1(\gamma_2)\,H_A^2(\gamma_2)$, - where γ_2 equals a or b - still in the right-hand side of each equation, is positive when at degree 1, the sign before the integral of $u_1(x)\,H_A^1(x)$ in the right-hand side is itself positive, and it is negative otherwise.

Carrying on till degree n yields:

- Degree n:

$$E_u(f)-E_u(g)=\sum_{i=2}^{n}\operatorname{sgn}(i-1)\,u_{i-1}(\gamma_i)\,H_{A,n}^i(\gamma_i)+\operatorname{sgn}(n)\int_a^b u_n(x)\,H_{A,n}^n(x)\,\mathrm{d}x \tag{2}$$

where for all $i\in[n]\,,\operatorname{sgn}(i)=1-2\mathbb{1}_C(i)\,.$

The set $C\in 2^{[n]}$ is the outcome of Procedure AC hereafter, which allocates each i to either C or $\complement_{[n]}C$ and, therefore, determines the values of $\operatorname{sgn}(n)$ and $\operatorname{sgn}(i-1)$ in (2).

Procedure AC

- *For all* $i\in A$

$$\begin{cases} \textit{if } i-1 \in C, \textit{ then } i \in \complement_{[n]}C \\ \textit{if } i-1 \in \complement_{[n]}C \textit{ or } i=1, \textit{ then } i \in C \end{cases}$$

- *For all* $i \in \complement_{[n]}A$

$$\begin{cases} \textit{if } i-1 \in \complement_{[n]}C \textit{ or } i=1, \textit{ then } i \in \complement_{[n]}C \\ \textit{if } i-1 \in C, \textit{ then } i \in C \end{cases}$$

B Proof of Proposition 1

In an expected utility framework, $fR_{B,n}g$ is equivalent to the non-negativity of $E_u(f) - E_u(g)$ for all $u \in U_{B,n}$, as given in (2).

Note that Procedure AB and Procedure AC above are identical, C in AC standing for B in AB, so that $B = C$. An immediate consequence is that in (2), the signs of $\operatorname{sgn}(i-1)$ and $u_{i-1}(x)$, as well as those of $\operatorname{sgn}(n)$ and $u_n(x)$, are always opposite.

Sufficiency:

$$\left.\begin{array}{r} fD_{A,n}g \\ \text{and, if } n \geq 3, \\ \text{for all } i = 2, ...n-1: \\ H^i_{A,n}(\gamma_i) \leq 0 \end{array}\right\} \Rightarrow fR_{B,n}g.$$

By hypothesis, when $n \geq 3$, for all i between 2 and n–1, $H^i_{A,n}(\gamma_i) \leq 0$ on the right-hand-side of (2). Still by hypothesis, $H^n_{A,n}(x) \leq 0$ for all x on $[a, b]$ - which also means that at the particular value γ_n, $H^n_{A,n}(\gamma_n) \leq 0$.

Since the signs of $\operatorname{sgn}(i)$ and $u_i(x)$ are always opposite for all $u \in U_{B,n}$, $\sum_{i=2}^n \operatorname{sgn}(i-1)\, u_{i-1}(\gamma_i)$ $H^i_{A,n}(\gamma_i) \geq 0$. In the same way, $\operatorname{sgn}(n) \int_a^b u_n(x)\, H^n_{A,n}(x)\, \mathrm{d}x \geq 0$, on the right-hand-side of (2). And since the whole right-hand-side of (2) is non-negative, $E_u(f) - E_u(g) \geq 0$ for all $u \in U_{B,n}$, which amounts to $fR_{B,n}g$.

Necessity:

$$fR_{B,n}g \Rightarrow \begin{cases} fD_{A,n}g \\ \text{and, if } n \geq 3, \\ \text{for all } i = 2, ...n-1: \\ H^i_{A,n}(\gamma_i) \leq 0. \end{cases}$$

Assume there exists y such that $H^n_{A,n}(y) > 0$. As a result, since the signs of $\operatorname{sgn}(n)$ and $u_n(y)$ are always opposite, if $n \in B$ (resp., $n \in {}_{[n]}(B)$), a sufficiently high (resp., low) value of $u_n(y)$ would allow the right-hand-side of (2) to become negative. And, since $E_u(f) - E_u(g)$ would become negative, this would contradict the assumption that $fR_{B,n}g$. One concludes that such value y does not exist, so that, for all x on $[a, b]$, $H^n_{A,n}(x) \leq 0$, which means that $fD_{A,n}g$.

Since this non-positivity of $H^n_{A,n}(x)$ also holds for the the upper and lower bounds of x, and consequently for γ_n, $H^n_{A,n}(\gamma_n) \leq 0$ in (2).

Let us turn, now, to the left-hand part of the right-hand-side of (2), *i.e.*, $\sum_{i=2}^n \operatorname{sgn}(i-1)$ $u_{i-1}(\gamma_i)\, H^i_{A,n}(\gamma_i)$. Note that this left-hand part only exists when $n \geq 2$, and that we have just shown that it is non-negative for $n = 2$, since $H^n_{A,n}(\gamma_n) \leq 0$.

Focusing on the cases when $n \geq 3$, imagine that there exists a value k of i $(2 \leq i \leq n-1)$ for which γ_k would lead to $H^k_{A,n}(\gamma_k) > 0$. Here again, a high enough absolute value of $u_{k-1}(\gamma_k)$

would allow the right-hand-side of (2) to become negative, contradicting the non-negativity of $E_u(f) - E_u(g)$ for each $u \in U_{B,n}$, which amounts to $fR_{B,n}g$. Hence, such value k does not exist, and $H^i_{A,n}(\gamma_i) \leq 0$ for all i included between 2 and n–1. □

References

J. Bernoulli (1713). *Ars Conjectandi* [English translation: *The Art of Conjecturing*, Baltimore: Johns Hopkins University Press, 2006].

J. Caballé and A. Pomansky (1996). Mixed Risk Aversion, *Journal of Economic Theory*, 71, pp. 485-513.

L. Eeckhoudt and H. Schlesinger (2006). Putting Risk in its Proper Place, *American Economic Review*, 96, pp. 280-89.

P. Fishburn (1974). Convex Stochastic Dominance with Continuous Distribution Functions, *Journal of Economic Theory*, 7, pp. 143-58.

P. Fishburn (1976). Continua of Stochastic Dominance Relations for Bounded Probability, *Journal of Mathematical Economics*, 3, pp. 295-311.

M. J. Goovaerts, F. E. C. De Vylder and J. Haezendonck (1984). *Insurance Premiums*, Amsterdam: North-Holland.

Z. Guo (2012). Stochastic Dominance and Its Applications in Portfolio Management, *Working Paper*, Konstanz University.

J. Hadar, W. R. Russell (1969). Rules for Ordering Uncertain Prospect, *American Economic Review*, 59(1), pp. 25-34.

J. Hadar, W. R. Russell (1971). Stochastic Dominance and Diversification, *Journal of Economic Theory*, 3, pp. 288-305.

G. Hanoch, H. Levy (1969). Efficiency Analysis of Choices Involving Risk, *Review of Economic Studies*, 36(3), pp. 335-46.

W.H. Jean, B.P. Helms (1988). The Identification of Stochastic Dominance Efficient Sets by Moment Combination Orderings, *Journal of Banking and Finance*, 12, pp. 243-53.

J. Karamata (1932). Sur une Inégalité Relative aux Fonctions Convexes, *Publications Mathématiques de l'Université de Belgrade*, 1, pp. 145-48.

M. S. Kimball (1990). Precautionary Savings in the Small and in the Large, *Econometrica*, 58, pp. 53-73.

M. S. Kimball (1992). Precautionary Motives for Holding Assets, in P. Newman, M. Milgate, and J. Falwell (eds.), *New Palgrave Dictionary of Money and Finance*, vol. 3. London: MacMillan, pp. 158-61.

F. Lajeri-Chaherli (2004). Proper Prudence, Standard Prudence and Precautionary Vulnerability, *Economics Letters*, 82(1), pp. 29- 34.

M. Le Breton (1987). Stochastic Dominance: A Bibliographical Rectification and a Restatement of Whitmore's Theorem, *Mathematical Social Science*, 13, pp. 73-9.

H. Levy (2006). *Stochastic Dominance – Investment Decision Making under Uncertainty*, 2nd edition, New-York: Springer Verlag.

P. Muliere and M. Scarsini (1989). A Note on Stochastic Dominance and Inequality Measures, *Journal of Economic Theory*, 49, pp. 314-23.

Th. Post, M. Kopa (2013). General Linear Formulations of Stochastic Dominance Criteria, *European Journal of Operational Research*, 230(2), pp. 321-32.

P.-Ch. Pradier (2006). De Usu Artis Conjectandi in Jure : Quid de Oeconomia (Politica) ?, *Journ@l Electronique d'Histoire des Probabilités et de la Statistique*, 2, pp. 1-15.

J.P. Quirk, R. Saposnik (1962). Admissibility and Measurable Utility Functions, *Review of Economic Studies*, 29, pp. 142-46.

M. Rothschild, J. E. Stiglitz (1970). Increasing Risk: I. A Definition, *Journal of Economic Theory*, 2, pp. 225-43.

S. Sriboonchitta, W.-K. Wong, S. Dhompongsa, and H.T. Nguyen (2010). *Stochastic Dominance and Applications to Finance, Risk and Economics*, Boca Raton: Chapman and Hall.

L. Tesfatsion (1976). Stochastic Dominance and the Maximization of Expected Utility, *Review of Economic Studies*, 43(2), pp. 301-15.

L. Thorlund-Petersen (2001). Third-Degree Stochastic Dominance and Axioms for a Convex Marginal Utility Function, *Mathematical Social Science*, 41, pp. 167-99.

G. A. Whitmore (1970). Third-Degree Stochastic Dominance, *American Economic Review*, 60(3), pp. 457-59.

K. Zaras (1989). Dominances Stochastiques pour Deux Classes de Fonctions d'Utilité, *Recherche Opérationnelle / Operations Research*, 23(1), pp. 57-65.