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Abstract

The recent financial crisis has highlighted the necessity to introduce mixtures of probability distributions in order to improve the estimation of asset returns and in particular to better take account of risks. Since Pearson (1894), these mixtures have been intensively used in many scientific fields since they provide very convenient mathematical tools to examine various statistical data and to approximate many probability distributions. They are typically introduced to model the choice of probability distributions among a given parametric family. The coefficients of the mixture usually correspond to the relative frequencies of each possible parameter. In this framework, we examine the single-period portfolio choice model, which has been addressed in the partial equilibrium framework, by Brennan and Solanki (1981), Leland (1980) and Prigent (2006). We consider an investor who wants to maximize the expected utility of the value of his portfolio consisting of one risk-free asset and one risky asset. We provide and analyze the solution for log return with mixture distributions, in particular for the mixture Gaussian case. The optimal portfolio is characterized for arbitrary utility functions. Our results show that mixture of distributions can have significant implications on the portfolio management.

1 Introduction

The recent financial crisis has highlighted the necessity to enhance the estimation of observed returns to better take account of risks and improve the estimation of asset returns. In this sense, introducing mixtures of probability distributions might help to achieve these aims (see McLachlan and Peel (2000) for definitions and properties of mixture models). The mixture distributions have been widely used in finance. For example, in the case of a finite mixture of Gaussian distributions, they could price standard and exotic options. Ritchey (1990) proved that the risk-neutral density of options could be modeled by a mixture of lognormal densities. Ryden et al. (1998) suggest to introduce hidden Markov chain to model daily return series, which leads immediately to mixture models. In a dynamic and finite mixture setting, Bellalah and Prigent (2002) provide an extension of the standard Black and Sholes models to price non-standard and exotic options and analyze the smile effect. Many others studies uses normal mixture returns to model excess kurtosis and to take account of the random volatility as in Alexander and Narayanan (2009). The literature characterizing empirical distributions discusses the utility of such models to fit financial data (see Bellalah and Lavielle, 2002; Hentati and Prigent, 2011) and local volatility (see Brigo et al. 2002; Alexander, 2004).

In this paper, we examine the single-period portfolio choice model[1], in the presence of Gaussian mixture log return distributions. We consider an investor who wants to maximize the expected utility of his terminal wealth, in a static way[2]. The value of the portfolio corresponds to a linear combination of some specified portfolio of common assets. We provide and analyze the solution for log return with mixture distributions, in particular for the mixture Gaussian case. The optimal portfolio is characterized for arbitrary utility functions.

Section 2 provides definitions and empirical examples of such Gaussian mix-

[1] The optimal positioning problem has been addressed in the partial equilibrium framework, by Brennan and Solanki (1981) and by Leland (1980).

[2] Due to practical constraints (liquidity, transaction costs...), financial portfolios are discretely rebalanced. For example, the portfolio is rebalanced monthly.

ture distributions for both an equity index (the MSCI world index) and a hedge fund index (the HFRX global index).

In Section 3, the optimal portfolio is determined and analyzed. The result is detailed in particular for CRRA utility functions. We emphasize the comparison between the optimal solution corresponding to the standard Gaussian case and the optimal portfolio in the presence of a Gaussian mixture. Finally, Section 4 concludes.

2 Gaussian mixtures

Many studies argue that a three Gaussian mixture is a good approximation of the empirical distribution: Melick and Thomas (1997) show that such mixture distribution is a very convenient tool to fit crude oil prices during the Golf's war; Bellalah and Lavielle (2002) prove also that, for the main equity financial indices, a three Gaussian mixture is a good approximation of the empirical distribution. The estimation of the mixture parameters has been examined for example by Peters and Walker (1978), Redner and Walker (1984), Basford and McLachlan (1985, 1988) and Leroux (1992). Their methods are usually based on the local ML estimation with consistent sequences of local maximizers.

2.1 Definitions and general properties

Suppose that each observation corresponds to a random vector $(X_1, ..., X_n)$, with respective cdf $(F_1, ..., F_n)$. Suppose, for example, that each variable X_i has a Gaussian distribution with mean m_i and variance-covariance matrix Σ_i. Denote $\theta_i\,(m_i, \Sigma_i)$ and $\lambda_i$the i-weight of the mixture. Let Φ the global mixture parameter:

$$\Phi = (\lambda_1, .., \lambda_n; \theta_1, .., \theta_n) \tag{1}$$

Then, the pdf corresponding to this mixture distribution is given by:

$$f\,(x, \Phi) = \sum_{i=1}^{n} \lambda_i f\,(x, \theta_i)\,, \tag{2}$$

where $f(x,\Phi)$ denotes the pdf of the multivariate Gaussian distribution $\mathcal{N}[m,\Sigma]$. The weighting system $(\lambda_i)_i$ corresponds to a convex combination. We have:

$$\sum_{i=1}^{n}\lambda_i = 1 \text{ and } \forall i \in \{1,..,n\}, \lambda_i > 0 \tag{3}$$

One explanation of such mixture is the following one: let Y be a discrete random variable with probability distribution defined by:

$$P(Y=i) = \lambda_i, for\ i = 1,...,n \tag{4}$$

Suppose that the conditional distribution of the vector X knowing Y is given by:$\mathcal{L}_X^{Y=i} = \mathcal{N}[m_i,\Sigma_i]$

Then, we deduce that the pdf of X satisfies: for any $x \in R^n$,

$$f_X(x) = \sum_{i=1}^{n}\lambda_i \frac{1}{\sqrt{(2\pi)^d |\Sigma_i|}} \exp\left\{-\frac{1}{2}(x-m_i)^T \Sigma_i^{-1}(x-m_i)\right\}. \tag{5}$$

Therefore, we get a Gaussian mixture with global parameter $\Phi = \{\lambda_i, m, \Sigma\}_{i=1}^{n}$ since for all $i = 1,...,n$, $\lambda_i > 0$ and $\sum_{i=1}^{n}\lambda_i = 1$.

An infinite mixture distribution corresponds to a pdf given by:

$$f(x,\Phi) = \int f(x,\Phi)\, g(y)\, dy,$$

where g(.) itself is a pdf. Suppose for example that the conditional distribution of the vector X knowing Y is given by:$\mathcal{L}_X^{Y=y} = \mathcal{N}[m_y,\Sigma_y]$

Then, we deduce that the pdf of X satisfies: for any $x \in R^n$,

$$f_X(x) = \int \frac{1}{\sqrt{(2\pi)^d |\Sigma_y|}} \exp\left\{-\frac{1}{2}(x-m_y)^T \Sigma_y^{-1}(x-m_y)\right\} g(y)\, d(y), \tag{6}$$

where $g(.)$ is the pdf of Y. Therefore, X has a Gaussian mixture distribution.

2.2 Empirical illustrations

To illustrate Gaussian mixtures, we use the weekly local MSCI world from December 1993 and August 2013 and the weekly HFRX global index, covering the period from January1998 until August 2013. To determine the mixture parameters, we apply the Expected Maximization (EM) algorithm based on Dempster et al. (1977). We investigate two cases: the two mixture distribution and the three mixture one.Next table provides the estimation of the mixture parameters for both financial indices. In all the cases, there exists at least a Gaussian distribution with negative mean and another one with positive mean. For the three mixture case, one explanation is that there exist three regimes: the first one corresponds to potential significant losses (for example, due to a financial crash), the second one to standard evolution of prices and finally, the third one to potential rises of the indices.

	MSCI word		
	Estimated weights of GM	Estimated mean vector of GM	Estimated variance matrice of GM
Normal distribution		0.000822807	0.000516511
2GM	0.6710	0.0041	1.97e-004
	0.3290	-0.0059	0.0011
3GM	0.6266	0.0024	2.4635e-004
	0.1593	0.0085	0.0017
	0.2141	0.0031	3.164e-004
	HFRI global index		
	Estimated weights of GM	Estimated mean vector of GM	Estimated variance matrice of GM
2GM	0.8449	0.0013	2.0616e-005
	0.1551	0.0015	4.7588e-004
3GM	0.0697	-0.0076	0.0026
	0.4022	0.0014	3.4194e-004
	0.5280	0.0018	2.4056e-005

Table 1: Gaussian mixture estimates for the MSCI world and HFRX global indices

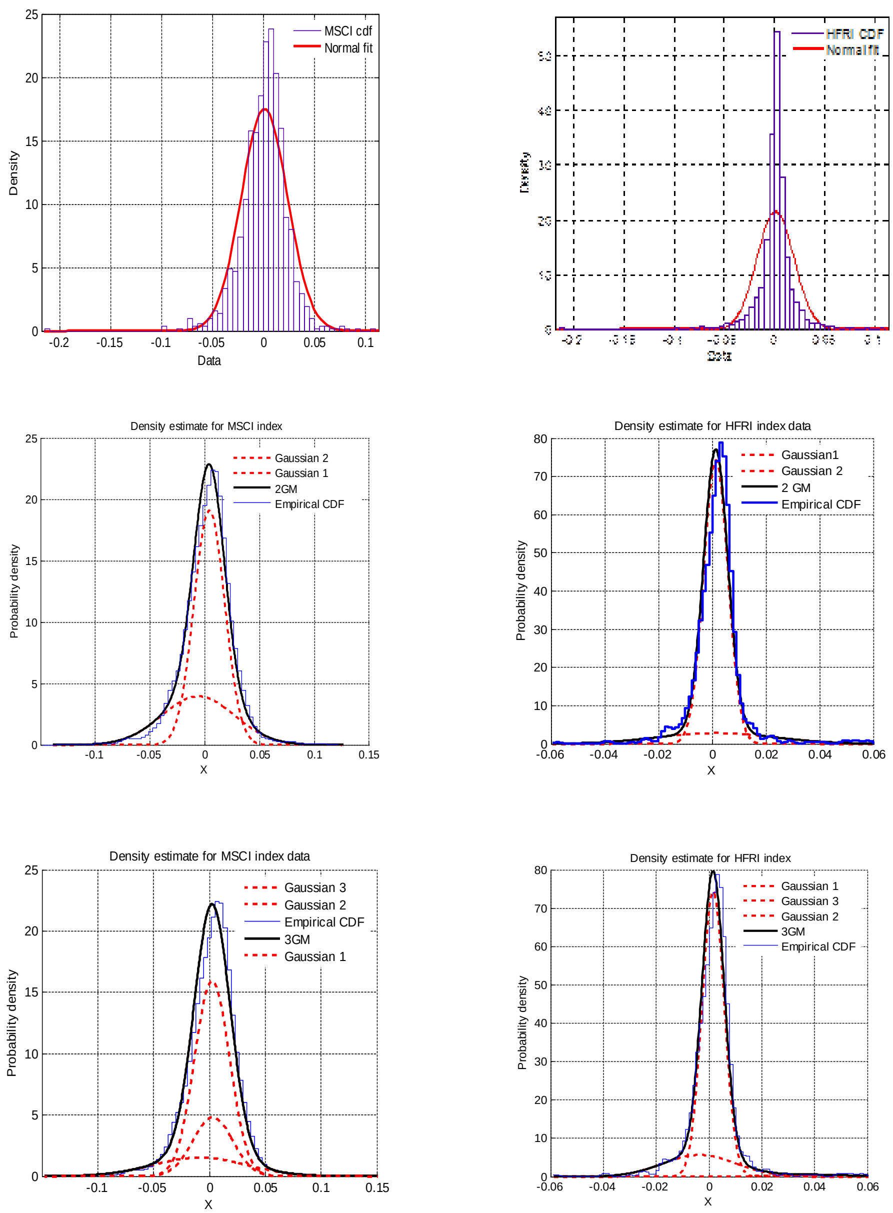

Figure 1. Gaussian mixtures (MSCI and HFRX indices)

2.3 Portfolio optimization

2.3.1 Buy-and-hold strategy

In what follows, we assume that the risky logreturn has a finite Gaussian mixture distribution. Its pdf is equal to:

$$f_X(x) = \sum_{i=1}^{n} \lambda_i, \tag{7}$$

where f_i is the pdf of the distribution $\mathcal{N}[m_i, \sigma_i]$ .Denote $\mu_i = m_i + \frac{\sigma_i}{2}$.. In what follows, we assume that the sequence $(\mu_i)_i$ is increasing, which is equivalent to the assumption that the expected returns $\int e^x f_i(x)\,dx$ are increasing.

The investor maximizes his expected utility:

$$Max_{ws} E[U[V_T]],$$

where V_T denotes the portfolio value at maturity T .We have:

$$V_T = V_0 \times \left(e^{rT} + w_s\left(e^{X_T} - e^{rT}\right)\right)$$

The first-order condition implies:

$$E\left[U'(V_T)\left(e^{X_T} - e^{rT}\right) = 0\right]$$

which is equivalent to:

$$\sum_{i=1}^{n} \int \lambda_i U'\left[V_0 \times \left(e^{rT} + w_s\left(e^{x} - e^{rT}\right)\right)\right]\left(e^{x} - e^{rT}\right) f_i(x)\,dx = 0 \tag{8}$$

We illustrate how the optimal solutions corresponding respectively to the Gaussian case and the Gaussian mixture case may differ. We assume that both probability distributions have same expectation and variance. Consider for instance the three Gaussian mixture case, usually observed on main equity indices for monthly logreturns. In that case, if the expectation is equal to α and the variance to σ^2 , then we have necessarily:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 1 \\ \lambda_1 e^{c} + \lambda_2 e^{\mu_2} + \lambda_3 e^{\mu_3} &= \alpha \\ \lambda_1 e^{2\mu_1}\left(e^{\sigma_1^2}-1\right) + \lambda_2 e^{\mu_2}\left(e^{\sigma_2^2}-1\right) + \lambda_3 e^{\mu_3}\left(e^{\sigma_3^2}-1\right) &= \sigma^2 \end{aligned}$$

with $\alpha > 0$, $\lambda_i \geq 0$.

Denote:

$$S_i^2 = e^{2\mu_i}\left(e^{\sigma_i}-1\right)$$

From previous relations, we deduce the parameter values λ_2 and λ_3 as function of λ_1. Since we have:

$$\begin{aligned} \lambda_2 + \lambda_3 &= 1 - \lambda_1, \\ \lambda_2 e^{\mu_2} + \lambda_3 e^{\mu_3} &= \alpha - \lambda_1 e^{\mu_1} \end{aligned}$$

We get:

$$\begin{aligned} \lambda_2 &= \frac{\left(e^{\mu_3}-\alpha\right) - \lambda_1, \left(e^{\mu_3}-e^{\mu_1}\right)}{\left(e^{\mu_3}-e^{\mu_2}\right)} \\ -\lambda_3 &= \frac{-\left(e^{\mu_2}-\alpha\right) - \lambda_1, \left(e^{\mu_2}-e^{\mu_1}\right)}{\left(e^{\mu_3}-e^{\mu_2}\right)} \end{aligned} \tag{9}$$

Finally, the coefficient $\lambda_{1,}$is given by:

$$\lambda_1 = \frac{\sigma^2\left(e^{\mu_3}-e^{\mu_2}\right) - S_2^2\left(e^{\mu_3}-\alpha\right) + S_3^2\left(e^{\mu_2}-\alpha\right)}{S_1^2\left(e^{\mu_3}-e^{\mu_2}\right) - S_2^2\left(e^{\mu_3}-e^{\mu_1}\right) + S_3^2\left(e^{\mu_2}-e^{\mu_1}\right)} \tag{10}$$

Therefore, if there exists a solution w_s of Equation 8 in [0,1], then we can apply the implicit functions theorem. Denote:

$$F\left(\lambda_1, w_s\right) = \sum_{i=1}^{3} \int \lambda_i U'\left[V_0 \times \left[V_0 \times \left(e^{rT} + w_s\left(e^{x}-e^{rT}\right)\right)\right]\left(e^{x}-e^{rT}\right) f_i\left(x\right) dx\right], \tag{11}$$

with λ_2 and λ_3 as functions of λ_1 , from Equation 9 (see Appendix for the bounds on λ_1).

We deduce the sensitivity of the optimal weight $ws*$ invested on the risky asset since we have:

$$\frac{\partial w_S^*}{\partial \lambda_1} = - \left[\frac{\partial F}{\partial \lambda_1} \right] \left[\frac{\partial F}{\partial w_S^*} \right] \quad (12)$$

Note that the second order derivative of the utility function is negative, since we assume that the investor is risk averse, so that his utility function is concave. Thus, $\left[\frac{\partial F}{\partial w_S^*} \right]$ is negative, which implies that w_S^* is a decreasing function of the weight λ_1 if and only if $\left[\frac{\partial F}{\partial \lambda_1} \right]$ is negative. This latter condition indicates if the investor prefers (or not) to invest on mixture distributions that overweight the two Gaussian distributions with higher exponential expectations.

To study the sign of $\left[\frac{\partial F}{\partial \lambda_1} \right]$, we note that:

$$\frac{\partial}{\partial \lambda_1} \left[\sum_{i=1}^{3} \int \lambda_i f_i \left(x \right) \right] = \frac{1}{\left(e^{\mu_3} - e^{\mu_2} \right)} g \left(x \right),$$

where

$$g \left(x \right) = \left(e^{\mu_3} - e^{\mu_2} \right) f_1 \left(x \right) - \left(e^{\mu_3} - e^{\mu_1} \right) f_2 \left(x \right) + \left(e^{\mu_2} - e^{\mu_1} \right) f_3 \left(x \right)$$

Then, the sign of $\left[\frac{\partial F}{\partial \lambda_1} \right]$ is the same as:

$$\int U' \left[V_0 \times \left(e^{rT} + w_s^* \left(e^x - e^{rT} \right) \right) \right] \left(e^x - e^{rT} \right) g \left(x \right) dx,$$

which depends mainly on the parameters of the three Gaussian distributions.

2.3.2 Numerical illustrations

To illustrate previous results, we use data the weekly local MSCI world from from December 1993 and August 2013 and the weekly HFRX global index, covering the period from January1998 until August 2013 (see Section 2).

We consider an investor with CRRA utility function given by , where denotes the relative risk aversion. Results for the optimal weight invested on the risky asset are displayed in Table 2.

Relative Risk Aversion	MSCI Gaussian case	MSCI Mixture case	HFRI Mixture case	HFRI Mixture case
0.5	100%	100%	100%	100%
2	100%	100%	100%	100%
5	100%	100%	60%	100%
10	60%	100%	30%	100%
20	30%	100%	15%	100%

Table 2: Optimal risky asset weight for the MSCI world and HFRX global indices

For both cases, we note that taking account of mixture models leads to higher investment on the risky asset. A similar result is also true for CARA utility $U(x) = -ae^{-ax}($ with $a > 0)$ and for utility with loss aversion as in Kahneman and Tversky (1992) $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $x < 0$ and $\frac{-(-x)^{1-\delta}}{1-\gamma}$ for $x < 0$ with $x < \gamma < 1$ and $\delta > 1)$. Such empirical examples show that, for a given utility function, there exists significant differences for the optimal portfolio when mixtures are taken into account, even if return expectations and variances are equal.

3 Conclusion

Using Gaussian mixtures allows to fit well empirical distributions. This kind of probability law is commonly used in financial modelling, through finite (regime switching due to economic variables, for instance) or infinite mixture (Lévy processes, Arch type models...). We show in this paper how it can be possible to optimize a portfolio, in this framework, and in a static way, since portfolio rebalancing takes place in discrete time. The main conclusion is that optimal portfolios for standard Gaussian case and mixture model case can differ very significantly, even if the risky financial returns have same expectation and standard deviation. Therefore, the mean-variance criterion is a not convenient criterion in the presence of mixture of distributions.

Appendix: Conditions on weights (bounds on $\lambda_{1,}$)

From Equation 9, we examine now the positivity condition on the weights λ_i :

1. $0 \leq \lambda_1 \leq 1$

2. $0 \leq \lambda_2 \leq 1 \Longleftrightarrow 0 \leq \left(e^{\mu_3} - \alpha\right) - \lambda_1 \left(e^{\mu_3} - e^{\mu_1}\right) \leq e^{\mu_3} - e^{\mu_2}$

   This condition is equivalent to:

$$\frac{e^{\mu_2} - \alpha}{e^{\mu_3} - e^{\mu_1}} \leq \lambda_1 \leq \frac{e^{\mu_3} - \alpha}{e^{\mu_3} - e^{\mu_1}}$$

3. $0 \leq \lambda_3 \leq 1 \Longleftrightarrow 0 \leq -\left(e^{\mu_2} - \alpha\right) + \lambda_1 \left(e^{\mu_2} - e^{\mu_1}\right) \leq e^{\mu_3} - e^{\mu_2}$ with $\left(e^{\mu_2} - e^{\mu_1}\right) > 0$

   This condition is equivalent to:

$$\frac{e^{\mu_2} - \alpha}{e^{\mu_3} - e^{\mu_1}} \leq \lambda_1 \leq \frac{e^{\mu_3} - \alpha}{e^{\mu_2} - e^{\mu_1}}$$

   Consequently, the positivity condition on the weights λ_i is equivalent to:

$$Max\left(0, \frac{e^{\mu_2} - \alpha}{e^{\mu_3} - e^{\mu_1}}, \frac{e^{\mu_2} - \alpha}{e^{\mu_2} - e^{\mu_1}}\right) \leq \lambda_1 \leq Min\left(1, \frac{e^{\mu_3} - \alpha}{e^{\mu_3} - e^{\mu_1}}, \frac{e^{\mu_3} - \alpha}{e^{\mu_2} - e^{\mu_1}}\right)$$

   But, since $e^{\mu_1} < e^{\mu_2} < e^{\mu_3}$, we get:

   (a) If $\alpha > e^{\mu_2}, Max\left(0, \frac{e^{\mu_2} - \alpha}{e^{\mu_3} - e^{\mu_1}}, \frac{e^{\mu_2} - \alpha}{e^{\mu_2} - e^{\mu_1}}\right) = 0$

   If $\alpha > e^{\mu_3}, Max\left(0, \frac{e^{\mu_2} - \alpha}{e^{\mu_3} - e^{\mu_1}}, \frac{e^{\mu_2} - \alpha}{e^{\mu_2} - e^{\mu_1}}\right) = \frac{e^{\mu_2} - \alpha}{e^{\mu_2} - e^{\mu_1}}$

   (b) $Min\left(1, \frac{e^{\mu_3} - \alpha}{e^{\mu_3} - e^{\mu_1}}, \frac{e^{\mu_3} - \alpha}{e^{\mu_2} - e^{\mu_1}}\right) = \frac{e^{\mu_3} - \alpha}{e^{\mu_3} - e^{\mu_1}}$

Finally, the positivity condition on the weights λ_i is equivalent to:

$$\frac{\left(e^{\mu_2} - \alpha\right)^+}{e^{\mu_2} - e^{\mu_1}} \leq \lambda_1 \leq \frac{\left(e^{\mu_3} - \alpha\right)}{e^{\mu_3} - e^{\mu_1}}$$

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References

[1] References

[2] Alexander, C. (2004): Normal mixture diffusion with uncertain volatility: modelling short and long term smile effects. Journal of Banking and Finance, 28, 2957-2980.

[3] Alexander, C. and Narayanan, (2009): Option pricing with normal mixture returns: modelling excess kurtosis and uncertainty in volatility, ICMA Centre, University of Reading.

[4] Basford, K.E., and McLachlan, G.J. (1985): Likelihood estimation with normal mixture models. Appl. Statist, 34, 282-289.

[5] Basford, K.E., and McLachlan, G.J. (1988). Mixture Models: Inference and Applications to Clustering. New York: Marcel Dekker.

[6] Bellalah, M. and Lavielle, M. (2002): A simple decomposition of empirical distributions and its applications in asset pricing. Multinational Finance Journal, 6, 99-130.

[7] Bellalah, M. and Prigent, J.-L. (2002): Pricing standard and exotic options in the presence of a finite mixture of Gaussian distributions, International Journal of Finance, 13, 1974-2000.

[8] Brennan, M.J. and Solanki, R., (1981). Optimal portfolio insurance. Journal of Financial and Quantitative Analysis, 16, 3, 279-300.

[9] Brigo, D., Mercurio, F., (2002): Lognormal-mixture dynamics and calibration to market volatility smiles. International Journal of Theoretical and Applied Finance 5, 427-446.

[10] Carr, P. and Madan, D., (2001). Optimal positioning in derivative securities. Quantitative Finance, 1, 19-37.

[11] Dempster, A., Laird, N., and Rubin, D. (1977): Maximum-likelihood from incomplete data via the EM algorithm. J. R. Statist. Soc., B 39, 1-38.

[12] Hentati, R. and Prigent, J.-L. (2011): On the maximization of financial performance measures within mixture models. Statistics and Decisions, 28, 1001-1018.

[13] Leland, H.E., (1980). Who should buy portfolio insurance? Journal of Finance 35, 581-594.

[14] Leroux, B.G.(1992): Consistant estimation of mixing distribution, Annals of Statistics, 20: 1350-1360.

[15] McLachlan, G.J. and Peel, D. (2000): Finite Mixture Models. Wiley Series in Probability and Statistics. Wiley.