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Abstract. This work generalize and extend results obtained recently in [2] from the Banach spaces framework to the groups framework. We study abstract classes of functions monoids for the inf-convolution structure and give a complete description of the group of unit of such monoids. We then apply this results to obtain various versions of the Banach-Stone theorem for the inf-convolution structure in the group framework. We also give as consequence an algebraic proof of the Banach-Dieudonée theorem. Our techniques are based on a new optimization result.

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Contents

1 Introduction.

We proved recently in [2] a version of the Banach-Stone theorem for the inf-convolution structure. More precisely, if $(X, \|.\|)$ is a Banach space and $\mathcal{CL}^1(X)$ denotes the set of all 1-Lipschitz convex and bounded from below functions, then $(\mathcal{CL}^1(X), \oplus)$ is a commutative monoid having $e = \|.\|$ as identity element for the operation $\oplus$ of inf-convolution. We proved that this monoid equipped with a natural metric, completely determine the Banach structure of X. In [2] we used the Banach-Dieudonné theorem (See Theorem 9) which applies only in this convex framework.

In this article we establish general results in the the group framework instead of the Banach spaces and we handle more general monoids. The tool used in this paper is a new result of optimization. Our first motivation is to prove a new versions of the Banach-Stone theorem for the inf-convolution structure in the group framework. For this purpose, we are going to study and give a complete and explicit description of the group of unit of general and abstract class of monoids for the inf-convolution structure. Historically, the inf-convolution appeared as tool of functional analysis and optimization (See for instance the work of [7], [10], [5], [12]) but it turns out as we are going to reveal it in this article, that the inf-convolution also enjoys a remarkable algebraic properties. Recall that the Banach-Stone theorem asserts that the Banach structure of the space $(C(K), \|.\|_\infty)$ of continuous functions on a compact space K completely determine the topological structure of K. More precisely, the Banach spaces $(C(K), \|.\|_\infty)$ and $(C(L), \|.\|_\infty)$ are isometrically isomorphic if and only if the compact spaces K and L are homeomorphic. The Banach-Stone theorem has been extended on various directions and other structure are considered by authors like the Banach algebra structure or unital vector lattice structure. The literature being very rich on this questions, we send back to the reference [3] for a more complete history and examples of extensions (See also [1] for the Banach-Stone theorem for the Banach structure on abstract class of function spaces).

In all this paper, we assume that $(X, ., e_X)$ is a group (not necessarily abelian) denoted multiplicatively and having the identity element e_X. By $\mathcal{F}(X)$ we denote the set of all maps defined from X into $\mathbb{R}$ and bounded fom below. The inf-convolution operation on $\mathcal{F}(X)$ (See also Moreau [7], [8]) is defined by

$$\begin{aligned} (f \oplus g)(x) &= \inf_{yz=x} \{f(y) + g(z)\}. \\ &= \inf_{z\in X} \left\{f(xz^{-1}) + g(z)\right\}. \end{aligned}$$

In general $f \oplus g \neq g \oplus f$ if X is not assumed to be an abelian group.
Clearly, $(\mathcal{F}(X), \oplus)$ is a semigroup. If moreover X is an abelian group then $(\mathcal{F}(X), \oplus)$ is a commutative semigroup. The semigroup $\mathcal{F}(X)$ is equiped with the useful metric :

$$d(f, g) := \sup_{x\in X} \frac{|(f(x) - \inf_X f) - (g(x) - \inf_X g)|}{1 + |(f(x) - \inf_X) - (g(x) - \inf_X g)|} + |\inf_X f - \inf_X g|, \;\; \forall f, g \in \mathcal{F}(X).$$

The following formulas is always true

$$\inf_X (f \oplus g) = \inf_X f + \inf_X g; \;\; \forall f, g \in \mathcal{F}(X).$$

This guarantees in particular that $\mathcal{F}_0(X) := \{f \in \mathcal{F}(X) : \inf_X f = 0\}$ is a subsemigroup of $(\mathcal{F}(X), \oplus)$.

1.1 Motivation and example.

The proposition bellow is the kind of results that we wish to show in this article.

Definition 1 *Let (X, m) be a metric group. We say that (X, m) is a metric invariant group if the metric m is invariant i.e*

$$m(x, y) = m(ax, ay) = m(xa, ya) \ \ \forall x, y, a \in X.$$

If moreover (X, m) is complete we say that (X, m) is complete metric invariant group.

Remark 1 *Every Fréchet space is a complete metric invariant group. For example of a non abelian complete metric invariant group see Example 2.*

We denote by $(Lip_0(X), \oplus)$ the semigroup of all Lipschitz and bounded from below functions f defined on (X, m) such that $\inf_X f = 0$ and $Lip^1_0(X)$ the monoid included in $Lip_0(X)$ of all 1-Lipschitz map. the monoid $Lip^1_0(X)$ has the map $\varphi_m : x \mapsto m(x, e_X)$ as identity element. The symbol $\cong$ denotes "isometrically isomorphic".

We obtain the following version of the Banach-Stone theorem for the inf-convolution structure which say that the monoid $(Lip^1_0(X), \oplus)$ completely determine the complete metric invariant group (X, m).

Proposition 1 *Let (X, m) and (Y, m') be complete metric invariant group. Then the following assertion are equivalent.*

(1) $(X, m) \cong (Y, m')$ *as groups.*

(2) $(Lip^1_0(X), d) \cong (Lip^1_0(Y), d)$ *as monoids.*

(3) *There exits a semigroup isomorphism isometric* $\Phi : (Lip_0(X), d) \to (Lip_0(Y), d)$ *such that* $\Phi(\varphi_m) = \varphi_{m'}$.

The proof of this result is based on the following two arguments (and follows from the more general Theorem **??**):

(1) An isomorphism of monoids send the group of unit $\mathcal{U}(Lip^1_0(X))$ of $Lip^1_0(X)$ on the group of unit $\mathcal{U}(Lip^1_0(Y))$ of $Lip^1_0(Y)$.

(2) The group of unit $\mathcal{U}(Lip^1_0(X))$ equipped with the metric d is isometrically isomorphic to $(X, \frac{m}{1+m})$. This is also equivalent to the fact that $\mathcal{U}(Lip^1_0(X))$ equiped with the metric d_∞ is isometrically isomorphic to (X, m), where $d_\infty(f, g) := \sup_{x \in X} \{|f(x) - g(x)|\} < +\infty$ for all $f, g \in \mathcal{U}(Lip^1_0(X))$.

1.2 The main algebraic results for the inf-convolution.

The study of abstract subsemigroups or submonoids of $\mathcal{F}(X)$ follows from the study of subsemigroups or submonoids of $\mathcal{F}_0(X)$.

Proposition 2 *$(\mathcal{F}(X), \oplus, d) \cong (\mathcal{F}_0(X) \times \mathbb{R}, \overline{\oplus}, d_1)$ as semigroups. Where $(f, t)\overline{\oplus}(g, s) := (f \oplus g, t + s)$ and $d_1((f, t); (g, s)) := d(f, g) + |t - s|$ for all $(f, t), (g, s) \in \mathcal{F}_0(X) \times \mathbb{R}$.*

Our result in this paper also applies for general monoids included in $\mathcal{F}_0(X)$. Let $M_{0,\varphi}(X)$ be an abstract monoid of $\mathcal{F}_0(X)$ and having φ as identity element, then φ is in particular an idempotent element i.e $\varphi \oplus \varphi = \varphi$. We wonder then if the result obtained in Proposition 1 hold for the abstract class of monoid $M_{0,\varphi}(X)$. The answer is affirmative for idempotent element φ satisfying the condition : $\varphi(x) = \varphi(x^{-1}) = 0 \Leftrightarrow x = e_X$. This motivates the following definition. Note that every monoid $M_{0,\varphi}(X)$ having φ as identity element is a submonoid of the following formal monoid

$$\mathcal{F}_{0,\varphi}(X) := \{f \in \mathcal{F}_0(X) : f \oplus \varphi = \varphi \oplus f = f\}.$$

Let us remark that since $\inf_X(f \oplus g) = \inf_X f + \inf_X g;\ \forall f, g \in \mathcal{F}(X)$, then every idempotent element of $\mathcal{F}(X)$ belongs necessarily to $\mathcal{F}_0(X)$ i.e $\varphi \oplus \varphi = \varphi \Rightarrow \varphi \geq 0 = \inf_X \varphi$.

Definition 2 *Let $\varphi \in \mathcal{F}(X)$. We say that φ is a remarkable idempotent if φ is an idempotent element and satisfy the following two properties:*

(1) $\varphi(xy) = \varphi(yx)$ pour tout $x, y \in X$ (Always true if X is commutative).

(2) $\varphi(x) = \varphi(x^{-1}) = 0 \Leftrightarrow x = e_X$.

We have the following more explicit characterization of remarkable idempotent (see section 3.).

Proposition 3 *Let $\varphi \in \mathcal{F}(X)$. Then, φ is remarkable idempotent if and only if φ satisfay :*

(1) $\varphi(xy) = \varphi(yx)$ for all $x, y \in X$.

(2) $\varphi(x) = \varphi(x^{-1}) = 0 \Leftrightarrow x = e_X$.

(3) $\varphi(xy) \leq \varphi(x) + \varphi(y)$ pour tout $x, y \in X$ (i.e φ is subadditive).

For each remarkable idempotent φ we can associate in a canonical way the metric $\Delta_{\infty,\varphi}$ on X defined by $\Delta_{\infty,\varphi}(x, y) := \max(\varphi(xy^{-1}), \varphi(yx^{-1}))$. We denote by $(\overline{X}, \overline{\Delta}_{\infty,\varphi})$ the group completion of $(X, \Delta_{\infty,\varphi})$. We denote by $\overline{\varphi}$ the unique extension of φ to $(\overline{X}, \overline{\Delta}_{\infty,\varphi})$ since φ is 1-Lipchitz for the metric $\Delta_{\infty,\varphi}$ by subadditivity. Note that $(\overline{X}, \overline{\Delta}_{\infty,\varphi})$ $=(\overline{X}, \Delta_{\infty,\overline{\varphi}})$ and that $\overline{\varphi}$ is also a remarkable idempotent. We need the following set which is a generalization of the set of 1-Lipschitz functions :

$$Lip^1_{0,\varphi}(X) := \left\{f \in \mathcal{F}_0(X) : f(x) - f(y) \leq \varphi(xy^{-1});\ \forall x, y \in X\right\}.$$

Since $\varphi(xy^{-1}) \leq \Delta_{\infty,\varphi}(x, y)$ and $\Delta_{\infty,\varphi}$ is a metric invariant then $Lip^1_{0,\varphi}(X)$ is a subset of $Lip^1_0(X)$ of all 1-Lipschitz map f on $(X, \Delta_{\infty,\varphi})$ such that $\inf_X f = 0$.

We have the following useful identification between the formal monoid $\mathcal{F}_{0,\varphi}(X)$ and the more explicit set $Lip^1_{0,\varphi}(X)$.

Proposition 4 *Let $\varphi \in \mathcal{F}(X)$ be a remarkable idempotent. Then $\mathcal{F}_{0,\varphi}(X) = Lip^1_{0,\varphi}(X)$ and so $Lip^1_{0,\varphi}(X)$ is a monoid having φ as identity element.*

Remark 2 *We recover from the above proposition the monoid $(Lip^1_0(X), \oplus)$ mentioned in the previous section from $\mathcal{F}_{0,\varphi_m}(X)$ where $\varphi_m : x \mapsto m(x, e_X)$ which is a remarkable idempotent.*

Proposition 5 *For every remarkable idempotent* $\varphi \in \mathcal{F}(X)$ *and every monoid* $M_{0,\varphi}(X)$ *we have that* $M_{0,\varphi}(X)$ *is a submonoid of the monoid* $Lip^1_{0,\varphi}(X)$*.*

The previous proposition explains that the study of abstract monoids of $\mathcal{F}_0(X)$ ensues from the study of the monoid $Lip^1_{0,\varphi}(X)$.

A. The group of unit.

Let us announce now our main algebraic result. We denote by $\mathcal{U}(Lip^1_{0,\varphi}(X))$ the group of unit of the monoid $(Lip^1_{0,\varphi}(X), \oplus)$.

Theorem 1 *Let* $\varphi \in \mathcal{F}(X)$ *a remarkable idempotent. Then,*

$$(\mathcal{U}(Lip^1_{0,\varphi}(X)), d_\infty) \cong (\overline{X}, \overline{\Delta}_{\infty,\varphi})$$

as groups. This is also equivalent to

$$(\mathcal{U}(Lip^1_{0,\varphi}(X)), d) \cong (\overline{X}, \frac{\overline{\Delta}_{\infty,\varphi}}{1 + \overline{\Delta}_{\infty,\varphi}}).$$

Since the completion of metric spaces is unique up to isometry, the following corollary gives an alternative way to considerer the completion of group metric invariant.

Corollary 1 *Let* (X, m) *a group metric invariant. Then*

$$(\overline{X}, \overline{m}) \cong (\mathcal{U}(Lip^1_0(X)), d_\infty).$$

Let us characterize now the group of unit of abstract monoid $M_{0,\varphi}(X)$ of $\mathcal{F}_0(X)$.

Corollary 2 *Let* $\varphi \in \mathcal{F}(X)$ *be a remarkable idempotent. Let* $M_{0,\varphi}(X)$ *be an abstract monoid of* $\mathcal{F}_0(X)$ *having* φ *as identity element. Then the group of unit* $\mathcal{U}(M_{0,\varphi}(X)), d)$ *of* $M_{0,\varphi}(X)$ *is isometrically isomorphic to a subgroup of* $(\overline{X}, \frac{\overline{\Delta}_{\infty,\varphi}}{1+\overline{\Delta}_{\infty,\varphi}})$*.*

B. The Banach-Stone theorem.

We obtain now the following general version of the Banach-Stone theorem for the inf-convolution structure.

Theorem 2 *Let* X *and* Y *be tow groups and let* $\varphi \in \mathcal{F}(X)$ *and* $\psi \in \mathcal{F}(Y)$ *be two remarkable idempotents. Then* $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Rightarrow (6)$*. If moreover we assume that* φ *and* ψ *are symetric (i.e* $\varphi(x) = \varphi(x^{-1})$ *and* $\psi(y) = \psi(y^{-1})$ *for all* $x \in X$ *and all* $y \in Y$*), then* $(1) - (6)$ *are equivalent.*

(1) *There exist a group isomorphism* $T : \overline{X} \to \overline{Y}$ *such that* $\overline{\psi} \circ T = \overline{\varphi}$*.*

(2) *There exist a semigroup isomorphism isometric* $\Phi : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ *sush that* $\Phi(0) = 0$ *and* $\Phi(\overline{\varphi}) = \overline{\psi}$*.*

(3) *There exist a semigroup isomorphism isometric* $\Phi : \mathcal{F}_0(\overline{X}) \to \mathcal{F}_0(\overline{Y})$ *sush that* $\Phi(\overline{\varphi}) = \overline{\psi}$*.*

(4) $(Lip^1_{0,\overline{\varphi}}(\overline{X}), d) \cong (Lip^1_{0,\overline{\psi}}(\overline{Y}), d)$ *as monoids.*

(5) $(Lip^1_{0,\varphi}(X), d) \cong (Lip^1_{0,\psi}(Y), d)$ *as monoids.*

(6) $(\overline{X}, \overline{\Delta}_{\infty,\varphi}) \cong (\overline{Y}, \overline{\Delta}_{\infty,\psi})$ *as groups.*

1.3 The main optimization results for the inf-convolution.

The algebraic main results of the previous sections follows from the following optimization result which applies on a general group metric invariant (not necessarily abelian). This result have many of other applications of optimization in particular for the resolution of the inf-convolution equations.

Definition 3 *Let (X, m) be a metric space, we say that a function f has a strong minimum at $x_0 \in X$, if $\inf_X f = f(x_0)$ and $m(x_n, x_0) \to 0$ whenever $f(x_n) \to f(x_0)$. A strong minimum is in particular unique.*

Theorem 3 *Let (X, m) be a complete metric invariant group with the identity element e_X. Let f and g be two lower semi continuous functions on (X, m). Suppose that the map $x \mapsto f \oplus g(x) + f \oplus g(x^{-1})$ has a strong minimum at e_X and $f \oplus g(e_X) = 0$. Then there exists $z_0 \in X$ such that :*

(1) *the map $\eta : z \to f(z^{-1}) + g(z)$ has a strong minimum at $z_0 \in X$ (we say that $f \oplus g(e_X)$ is attained strongly at z_0).*

(2) *$f(x) \geq f \oplus g(xz_0) + f(z_0^{-1})$ and $g(x) \geq f \oplus g(z_0^{-1}x) + g(z_0)$ for all $z \in X$.*

The following result shows that a strong linear perturbation of the convolution $f \oplus g$ at some point x_0 of two lower semi continuous functions f and g leads to a strong perturbation of f and g with the same perturbation on respectively some points x_1 and x_2 such that $x_1 x_2 = x_0$.

Corollary 3 *Let (X, m) be a complete metric invariant group with the identity element e_X. Let $p : X \to \mathbb{R}$ be a group morphism and f and g be two lower semi continuous functions on (X, m). Suppose that the map $x \mapsto f \oplus g(x) - p(x)$ has a strong minimum at x_0 , then there exists $z_0 \in X$ such that*

(1) *the map $\eta : z \to f(x_0 z^{-1}) + g(z)$ has a strong minimum at $z_0 \in X$ i.e $f \oplus g(x_0)$ is attained strongly at z_0.*

(2) *$f - p$ has a strong minimum at $x_0 z_0^{-1}$ and $g - p$ has a strong minimum at z_0.*

1.4 Organization of the paper.

This article is organied as follow. In section 2. we give some examples of complete metric invariant group, remarkable idempotent and monoids for the inf-convolution structure. In section 3. we give the proof of our main optimization result Theorem 5 (Theorem 3 in the introduction). In section 4. we give several algebraic properties of the inf-convolution structure and the proof of our main algebraic result Theorem 6 (Theorem 1 in the introduction). In section 5. We give various versions of the Banach-Stone theorem and the proof of the main result of this section Theorem 7 (Theorem 2 in the introduction). Finally in section 6. We give an algebraic proof of the well know Banach-Dieudonné theorem (See Theorem 9).

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2 Examples.

A. Complete metric invariant groups.

Exemples 1 (Abelian group case).

(1) *Every Fréchet space is complete metric invariant group. In particular every Banach space equiped with the metric associated the the norm is a complete metric invariant group.*

(2) *Let E be a set of finite cardinal and $\mathcal{P}(E)$ the set of all subset of E. The set $(\mathcal{P}(E), \Delta)$ is an abeliean group, where Δ is the symmetric difference between two sets : $A\Delta B = (A \cup B) \setminus (A \cap B)$ for all $A, B \in \mathcal{P}(E)$. We denote by $|A|$ the cardinal of A. Then $(\mathcal{P}(E), m)$ is a complete metric group where m is the metric defined by $m(A, B) = \frac{|A\Delta B|}{|E|}$ on $\mathcal{P}(E)$.*

(3) *Every group X is complete metric invariant group for the discrete metric.*

Exemples 2 (Non abelian group case). *Let H be a real separable Hilbert space, $\mathcal{O}(H)$ be the orthogonal group on H and I be the identity operator. We denote by*

$$\mathcal{O}_c(H) := \{T \in \mathcal{O}(H) : I - T \text{ is a comact operator}\}$$

and

$$\mathcal{O}_h(H) := \{T \in \mathcal{O}(H) : I - T \text{ is a Hilbert-Shmidt operator}\}.$$

The metrics d_c and d_h as defined as follow : $d_c(T, S) = \|T - S\|_{op}$ and $d_h(T, S) = \|T - S\|_{HS}$ where $\|.\|_{op}$ is the norm operator and $\|.\|_{HS}$ is the Hilbert-Schmidt norm

$$\|A\|_{HS}^2 = \mathrm{Tr}|(A^* A)| := \sum_{i \in I} \|Ae_i\|^2$$

where $\|.\|$ is the norm of H and $\{e_i : i \in \mathbb{N}\}$ an orthonormal basis of H. This definition is independent of the choice of the basis.

***Theorem 4* (See [[11], Theorem 1.1])** *$(O_C(H), d_c)$ and $(O_h(H), d_h)$ are complete separable metric invariant (non abeliean) groups.*

B. Remarkable idempotent.

Exemples 3

(1) *Let (X, m) be a metric invariant group with the identity element e_X and $0 < \alpha \leq 1$. Then the map φ in the following cases is remarkable idempotent in $\mathcal{F}(X)$. In this cas φ is symetric $\varphi(x) = \varphi(x^{-1})$ for all $x \in X$.*

(a) the map $\varphi(x) := m(e_X, x)^\alpha$ for all $x \in X$.

(b) Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and sub-additive function having a strong minimum at 0 and such that $\chi(0) = 0$. We define φ as follow. $\varphi(x) := \chi(m(e_X, x)^\alpha)$.

(2) *Let $(X, \|.\|_X)$ be a real normed vector space. Let C be a convex bounded subset of X containing the origin. Then the Minkowski functional $\varphi_C(x) := \inf\{\lambda > 0 : x \in \lambda C\}$ for all $x \in X$ is remarkable idempotent.*

(3) *Let* $(X, \|.\|_X)$ *be a vector normed space , and* $K \subset X^*$ *be a convex weak-star closed and bounded such that int*$(K) \neq \emptyset$ *(int*(K) *denotes the interior of* K *for the norm topology). Then the support function defined by* $\sigma_K(x) := \sup_{p\in K} p(x)$ *is remarkable idempotent.*

(4) *Let* X *be any group with the identity element* e_X*. Then the map* φ_{e_X} *defined by* $\varphi_{e_X}(e_X) = 0$ *and* $\varphi_{e_X}(x) = 1$ *if* $x \neq e_X$ *is remarkable idempotent.*

C. Examples of monoids for the inf-convolution.

Exemples 4

(1) *Let* $(X, \|.\|_X)$ *be a Banach space and and* $K \subset X^*$ *be a convex weak-star closed and bounded such that int*$(K) \neq$ *emptyset. Let* $\mathcal{LC}(X)$ *the set of all bounded from below convex and Lipschitz functions on* X*. Then* $(M_{0,\sigma_K}(X), \oplus) := (\mathcal{LC}(X) \cap Lip_{0,\sigma_K}(X), \oplus)$ *is a monoid having* σ_K *as identity element. If* $K = B_{X^*}$ *then* $\sigma_K = \|.\|$ *and in this case we recover the monoid studied in [2].*

(2) *Let* (X, m) *be a complete metric invariant group with identity element* e_X *and let* $0 < \alpha \leq 1$ *and* $\varphi_m(x) = m^\alpha(e_x; x)$ *for all* $x \in X$*. Let* $Lip^\alpha(X)$ *be the set of all bounded from below,* α*-Hölder functions on* X *and* $Lip^\alpha(X)$ *is the set of all* α*-Hölder functions on* X *such that* $\inf_X f = 0$*. Then* $Lip^\alpha(X)$ *and* $Lip^\alpha(X)$ *are monoid having* φ_m *as identity element.*

(3) *Let* $IC(\mathbb{R}^n)$ *be the set of all inf-compact functions from* $\mathbb{R}^n$ *into* $\mathbb{R}$ *and let* $0 < \alpha \leq 1$*. Then* $(IC(\mathbb{R}^n), \oplus)$ *is a semigroup (See [7]) and* $IC(\mathbb{R}^n) \cap Lip^\alpha(\mathbb{R}^n)$ *is a monoid having* $\varphi = \|.\|^\alpha$ *as identity element.*

(4) *Let* X *be any algebraic group with the identity element* e_X *and let* m *be the discrete distance on* X *and* $\delta_{e_X}(x) = 0$ *if* $x = e_X$ *and take the value 1 otherwise. Let* $\mathcal{F}^1(X) = \{f : X \to \mathbb{R} : \sup_{x,y\in X} |f(x) - f(y)| \leq 1\}$ *and* $\mathcal{F}^1_0(X) := \{f : X \to [0, 1] : \inf_X f = 0\}$*. Then* $\mathcal{F}^1(X)$ *and* $\mathcal{F}^1_0(X)$ *are monoids having* δ_{e_X} *as identity element.*

3 The proof of the main optimization results.

Theorem 5 *Let* (X, m) *be a complete metric invariant group with the identity element* e_X*. Let* f *and* g *be two lower semi continuous functions on* (X, m)*. Suppose that the map* $x \mapsto f \oplus g(x) + f \oplus g(x^{-1})$ *has a strong minimum at* e_X *and* $f \oplus g(e_X) = 0$*. Then there exists* $z_0 \in X$ *such that :*

(1) *the map* $\eta : z \to f(z^{-1}) + g(z)$ *has a strong minimum at* $z_0 \in X$*.*

(2) $f(x) \geq f \oplus g(xz_0) + f(z_0^{-1})$ *and* $g(x) \geq f \oplus g(z_0^{-1}x) + g(z_0)$ *for all* $z \in X$*.*

Proof. (1) Let $(z_n)_n \subset X$ be such that for all $n \in \mathbb{N}^*$,

$$f \oplus g(e_X) \leq \quad f(z_n^{-1}) + g(z_n) \quad < f \oplus g(e_X) + \frac{1}{n}.$$

Since $f \oplus g(e_X) = 0$ then

$$0 \leq \quad f(z_n^{-1}) + g(z_n) \quad < \frac{1}{n}. \tag{1}$$

On the other handfor all $x, y \in X$,

$$f \oplus g(xy^{-1}) \leq f(x) + g(y^{-1}) \quad (2)$$
$$f \oplus g(yx^{-1}) \leq f(y) + g(x^{-1}). \quad (3)$$

By adding both inequalilies (2) and (3) above we obtain for all $x, y \in X$

$$f \oplus g(xy^{-1}) + f \oplus g(yx^{-1}) \leq \left(f(x) + g(x^{-1})\right) + \left(f(y) + g(y^{-1})\right). \quad (4)$$

By appllaying the above inequality with $x = z_n^{-1}$ and $y = z_m^{-1}$, we have

$$f \oplus g(z_n^{-1} z_m) + f \oplus g(z_m^{-1} z_n) \leq \left(f(z_n^{-1}) + g(z_n)\right) + \left(f(z_m^{-1}) + g(z_m)\right)$$

From our hypothesis we have that the map $z \to f \oplus g(z) + f \oplus g(z^{-1})$ has a stong minimum at e_X with $0 = f \oplus g(e_X) + f \oplus g(e_X^{-1})$. So from the above inequality and (1) we obtain

$$0 \leq f \oplus g(z_n^{-1} z_m) + f \oplus g(z_m^{-1} z_n) \leq \frac{1}{n} + \frac{1}{m}.$$

Thus $f \oplus g(z_n^{-1} z_m) + f \oplus g((z_n^{-1} z_m)^{-1}) \to 0$ when $n, m \to +\infty$ which implies that $m(e_X, z_n^{-1} z_m) \to 0$ or equivalently $m(z_n, z_m) \to 0$ since m is invariant. Thus the sequence $(z_n)_n$ is Cauchy in (X, m) and so converges to some z_0 since (X, m) is a complete metric space. By the lower semi-continuity of f and g, the continuity of $z \to z^{-1}$ and by using the formulas (1) we get

$$f(z_0^{-1}) + g(z_0) \leq 0 = f \oplus g(e_X).$$

On the other hand, by definition we have $f \oplus g(e_X) \leq f(z_0^{-1}) + g(z_0)$. Thus

$$f(z_0^{-1}) + g(z_0) = f \oplus g(e_X) = 0. \quad (5)$$

It follows that η has a minimum at z_0 (by definition we have $\inf_{z \in X} \eta(z) = f \oplus g(e_X)$). To see that η has a strong minimum at z_0, let $(x_n)_n$ be any sequence such that $f(x_n^{-1}) + g(x_n) \to \inf_{z \in X} \left\{f(z^{-1}) + g(z)\right\} = 0$. By applying (4) with $x = z_0^{-1}$ and $y = x_n^{-1}$ and the formulas (5) and (1) with the fact that $f \oplus g(z) + f \oplus g(z^{-1}) \geq 0$ for all $z \in X$, we obtain that $f \oplus g(z_0^{-1} x_n) + f \oplus g(x_n^{-1} z_0) \to 0$ which implies by hypothesis that $m(x_n, z_0) \to 0$. Thus z_0 is a strong minimum of η.

(2) Using the part (a) we have that $0 = f \oplus g(e_X) = f(z_0^{-1}) + g(z_0)$. We have

$$\begin{aligned} f \oplus g(z_0^{-1} x) &= \inf_{y \in X} \left\{f(z_0^{-1} x y^{-1}) + g(y)\right\} \\ &\leq f(z_0^{-1}) + g(x) \\ &= -g(z_0) + g(x). \end{aligned} \quad (6)$$

and

$$\begin{aligned} f \oplus g(x z_0) &= \inf_{y \in X} \left\{f(x z_0 y^{-1}) + g(y)\right\} \\ &\leq f(x) + g(z_0) \\ &= -f(z_0^{-1}) + f(x). \end{aligned} \quad (7)$$

This ends the proof of (2).■

Lemma 1 *Let (X, m) be a metric group with the identity element e_X and let $h : X \to \mathbb{R}$. Suppose that h has a strong minimum at e_X and $h(e_X) = 0$, then the map $x \mapsto h(x) + h(x^{-1})$ has a strong minimum at e_X and $h(e_X) = 0$.*

Remark 3 *The converse of the above proposition is not true in general (Take $h(x) = x + |x|$ on $X = (\mathbb{R}, +)$).*

Proof : Since h has a strong minimum at e_X and $h(e_X) = 0$ then $h(x) \geq h(e_X) = 0$ for all $x \in X$. So $h(x) + h(x^{-1}) \geq 0$. On the other hand, we have $h(e_X) + h(e_X^{-1}) = 2h(e_X) = 0$, and so $x \to h(x) + h(x^{-1})$ has a minimum at e_X. Let us show that e_X is a strong minimum for $x \to h(x) + h(x^{-1})$. Indeed, since $h \geq 0$, then we have,

$$0 \leq h(x) \leq h(x) + h(x^{-1})$$

for all $x \in X$. If $(z_n)_n$ is a sequence such that $h(z_n)+h(z_n^{-1}) \to \inf_{x \in X} \big(h(x) + h(x^{-1})\big) = 0$ then by the above inequalities we have that $h(z_n) \to 0$ which implies that $z_n \to e_X$ since h has a strong minimum at e_X. Thus $x \to h(x) + h(x^{-1})$ has a strong minimum at e_X and $h(e_X) = 0$.

Corollary 4 *Let (X, m) be a complete metric invariant group with the identity element e_X. Let $p : X \to \mathbb{R}$ be a group morphism and f and g be two lower semi continuous functions on (X, m). Suppose that the map $x \mapsto f \oplus g(x) - p(x)$ has a strong minimum at x_0 , then there exists $z_0 \in X$ such that*

(1) the map $\eta : z \to f(x_0 z^{-1}) + g(z)$ has a strong minimum at $z_0 \in X$ and $f(x_0 z_0^{-1}) + g(z_0) = f \oplus g(x_0)$. In particular $f \oplus g(x_0)$ is exact at x_0.

(2) $f - p$ has a strong minimum at $x_0 z_0^{-1}$ and $g - p$ has a strong minimum at z_0.

Proof. First, note that $f \oplus g(x) - p(x) = (f - p) \oplus (g - p)(x)$ for all $x \in X$ since p is a group morphism. Let us set $c := f \oplus g(x_0) - p(x_0)$ and $h : t \mapsto f \oplus g(x_0 t) - p(x_0 t) - c$. Then h has a strong minimum at e_X since by hypothesis $x \mapsto f \oplus g(x) - p(x)$ has a strong minimum at x_0. Let us denote by $\tilde{f} : t \mapsto f(x_0 t) - p(x_0 t) - c$ and $\tilde{g} : t \mapsto g(t) - p(t)$. Then we have that $\tilde{f}$ and $\tilde{g}$ are lower semi-continuous and $\tilde{f} \oplus \tilde{g} = h$. We deduce then that the map $x \mapsto \tilde{f} \oplus \tilde{g}(x)$ has a strong minimus at e_X. Thus by Lemma 1 we have that $x \mapsto \tilde{f} \oplus \tilde{g}(x) + \tilde{f} \oplus \tilde{g}(x^{-1})$ has a strong minimum at e_X and we can apply Theorem 5 to obtain the existence of some $z_0 \in X$ such that:

(1) the map $\eta : z \to \tilde{f}(z^{-1}) + \tilde{g}(z)$ has a strong minimum at $z_0 \in X$.

(2) $\tilde{f}(x) \geq \tilde{f} \oplus \tilde{g}(x z_0) + \tilde{f}(z_0^{-1})$ and $\tilde{g}(x) \geq \tilde{f} \oplus \tilde{g}(z_0^{-1} x) + \tilde{g}(z_0)$ for all $z \in X$.
Using the fact that p is a group morphism and by replacing $\tilde{f}$ and $\tilde{g}$ by their expression, we translate (1) and (2) respectively as follow

(1′) the map $\eta : z \to f(x_0 z^{-1}) + g(z)$ has a strong minimum at $z_0 \in X$.

(2′) $f(x) - p(x) \geq (f \oplus g(x) - p(x)) - (f \oplus g(x_0) - p(x_0)) + \big(f(x_0 z_0^{-1}) - p(x_0 z_0^{-1}\big)$ and $g(x) - p(x) \geq \big(f \oplus g(x_0 z_0^{-1} x) - p(x_0 z_0^{-1} x)\big) - (f \oplus g(x_0) - p(x_0)) + (g(z_0) - p(z_0)$, for all $x \in X$.
Using the fact that $x \mapsto f \oplus g(x) - p(x)$ has a strong minimum at x_0, this implies respectively

(1″) the map $\eta : z \to f(x_0 z^{-1}) + g(z)$ has a strong minimum at $z_0 \in X$.

(2″) $f(x) - p$ has a strong minimum at $x_0 z_0^{-1}$ and $g - p$ has a strong minimum at z_0.

4 The inf-convolution and algebra.

4.1 Properties and useful lemmas.

4.1.1 The semigroup $\mathcal{F}_0(X)$.

We have the following more explicit characterization of remarkable idempotent. The proof follows immediately from Lemma 2.

Proposition 6 *Let $\varphi \in \mathcal{F}(X)$. Then, φ is remarkable idempotent if and only if φ satisfay :*

(1) $\varphi(xy) = \varphi(yx)$ *for all* $x, y \in X$.

(2) $\varphi(x) = \varphi(x^{-1}) = 0 \Leftrightarrow x = e_X$.

(3) $\varphi(xy) \leq \varphi(x) + \varphi(y)$ *pour tout* $x, y \in X$ *(i.e φ is subadditive).*

Lemma 2 *Let X be a group and e_X its identity element. Suppose that $\varphi(e_X) = 0$. Then $\varphi \oplus \varphi = \varphi$ if and only if φ is sub-additive i.e $\varphi(xy) \leq \varphi(x) + \varphi(y)$ for all $x, y \in X$.*

Proof. ($\Rightarrow$) Suppose that $\varphi \oplus \varphi = \varphi$. Then

$$\begin{aligned} \varphi(xy) &= \inf_{z \in X} \left\{ \varphi(xyz^{-1}) + \varphi(z) \right\} \\ &\leq \varphi(y) + \varphi(x); \quad \forall x, y \in X. \end{aligned}$$

($\Leftarrow$) For the converse suppose that $\varphi(xy) \leq \varphi(x) + \varphi(y)$ for all $x, y \in X$. Then we have $\varphi(x) = \varphi((xz^{-1})z) \leq \varphi(xz^{-1}) + \varphi(z); \forall x, z \in X$. Taking the infinitum over $z \in X$ we get $\varphi(x) \leq \varphi \oplus \varphi(x)$ for all $x \in X$. Now

$$\begin{aligned} \varphi \oplus \varphi(x) &= \inf_{z \in X} \left\{ \varphi(xz^{-1}) + \varphi(z) \right\} \\ &\leq \varphi(x) + \varphi(e_X) \\ &= \varphi(x). \end{aligned}$$

Thus $\varphi \oplus \varphi = \varphi$.$\blacksquare$

Lemma 3 *Let $\varphi \in \mathcal{F}(X)$ be a remarkable idempotent. Then*

(1) *Then for all $f \in \mathcal{F}(X)$ we have $f \oplus \varphi = \varphi \oplus f$ (all elements $f \in \mathcal{F}(X)$ commutes with φ).*

(2) $Lip^1_{0,\varphi}(X) = \mathcal{F}_{0,\varphi}(X)$ *(in particular $(Lip^1_{0,\varphi}(X), \oplus, \varphi)$ is a monoid).*

(3) *Every elements f of $\mathcal{F}_{0,\varphi}(X)$ is 1-Lipschitz for the metric $\Delta_{\infty,\varphi}$.*

Proof. (1) Let us first proves that for all $f \in \mathcal{F}(X)$ we have $f \oplus \varphi = \varphi \oplus f$. Indeed, by using the fact that $\varphi(xy) = \varphi(yx)$ for all $x, y \in X$ with the following variable change $t = xy^{-1}$ we have for all $x \in X$,

$$\begin{aligned} f \oplus \varphi(x) &= \inf_{y \in X} \left\{ f(xy^{-1}) + \varphi(y) \right\} \\ &= \inf_{y \in X} \left\{ f(t) + \varphi(t^{-1}x) \right\} \\ &= \inf_{t \in X} \left\{ \varphi(xt^{-1}) + f(t) \right\} \\ &= \varphi \oplus f(x) \end{aligned}$$

(2) We prove that $\mathcal{F}_{0,\varphi}(X) \subset Lip^1_{0,\varphi}(X)$: Let $f \in \mathcal{F}_{0,\varphi}(X)$. Then by the definition of $\mathcal{F}_{0,\varphi}(X)$ we have $\varphi \oplus f = f \oplus \varphi = f$. We are going to prove that $f \in Lip^1_{0,\varphi}(X)$. Indeed, let $x, y \in X$ and let $(z_n)_n \subset X$ such that for all $n \in \mathbb{N}^*$

$$\varphi \oplus f(y) > \varphi(yz_n^{-1}) + f(z_n) - \frac{1}{n} \tag{8}$$

On the other hand ,

$$\varphi \oplus f(x) \leq \varphi(xz_n^{-1}) + f(z_n) \tag{9}$$

By combining (8) and (9) we have

$$\varphi \oplus f(x) \leq \varphi \oplus f(y) + \varphi(xz_n^{-1}) - \varphi(yz_n^{-1}) + \frac{1}{n} \tag{10}$$

Now using Lemma 2 we have $\varphi \oplus \varphi = \varphi$ and so we have

$$\begin{aligned}
\varphi(xz_n^{-1}) &= \varphi \oplus \varphi(xz_n^{-1}) \\
&= \inf_{t \in X} \left\{\varphi(xz_n^{-1}t^{-1}) + \varphi(t)\right\} \\
&\leq \varphi(xy^{-1}) + \varphi(yz_n^{-1}).
\end{aligned} \tag{11}$$

Combining (10) and (11) and sending n to $+\infty$ we obtain that

$$\varphi \oplus f(x) \leq \varphi \oplus f(y) + \varphi(xy^{-1}).$$

This shows that $\varphi \oplus f \in Lip^1_{0,\varphi}(X)$. But $\varphi \oplus f = f$, thus $f \in Lip^1_{0,\varphi}(X)$.

We prove now that $Lip^1_{0,\varphi}(X) \subset \mathcal{F}_{0,\varphi}(X)$: Let $f \in Lip^1_{0,\varphi}(X)$. From part (1) we have $f \oplus \varphi = \varphi \oplus f$. We are going to prove that $f \oplus \varphi = f$. By the definition of $Lip^1_{0,\varphi}(X)$ we have

$$f(x) \leq f(y) + \varphi(xy^{-1}); \forall x, y \in X.$$

Taking the infinimum over $y \in X$, we get $f(x) \leq f \oplus \varphi(x)$ for all $x \in X$. For the converse inequality we have

$$\begin{aligned}
f \oplus \varphi(x) &= \inf_{y \in X} \left\{f(y) + \varphi(xy^{-1})\right\} \\
&\leq f(x) + \varphi(e_X) \\
&= f(x).
\end{aligned}$$

Thus $f \oplus \varphi = \varphi \oplus f = f$ and so $f \in \mathcal{F}_{0,\varphi}(X)$.
(3) This part follows easily from the part (2) and the definition of $Lip^1_{0,\varphi}(X)$.$\blacksquare$.

Lemma 4 *Let $\varphi \in \mathcal{F}(X)$ be a remarkable idempotent and $\overline{\varphi}$ be the unique extension of φ to the completion $(\overline{X}, \overline{\Delta}_{\infty,\varphi})$. Then $(Lip^1_{0,\varphi}(X), d) \cong (Lip^1_{0,\overline{\varphi}}(\overline{X}), d)$ as monoids. More precisely, the map*

$$\begin{aligned}
\chi : (Lip^1_{0,\varphi}(X), \oplus, d) &\rightarrow (Lip^1_{0,\overline{\varphi}}(\overline{X}), \oplus, d) \\
f &\mapsto \overline{f} = \left[\bar{x} \mapsto \inf_{z \in X} \left\{\overline{\varphi}(\bar{x}z^{-1}) + f(z)\right\}\right]
\end{aligned}$$

is an isometric isomorphism.

Proof. It is easy to see that $\overline{f} \in Lip^1_{0,\overline{\varphi}}(\overline{X})$ for all $f \in Lip^1_{0,\varphi}(X)$ and that the map χ is well defined since if $f = g$ on X then clearly $\overline{f} = \overline{g}$ on $\overline{X}$. Observe that the restriction $f_{|X}$ of $\overline{f}$ to X coincide with $\varphi \oplus f$, by definition of $\overline{f}$. The map χ is injective since, if $\overline{f} = \overline{g}$ on $\overline{X}$ then by the restriction to X we obtain $\varphi \oplus f = \varphi \oplus g$. Thus $f = g$ since $Lip^1_{0,\varphi}(X) = \mathcal{F}_{0,\varphi}(X) := \{f \in \mathcal{F}_0(X) : f \oplus \varphi = \varphi \oplus f = f\}$ by Lemma 3. The map χ is surjective. Indeed, let $F \in Lip^1_{0,\overline{\varphi}}(\overline{X})$ and set $f = F_{|X}$ the restriction of F to X. Then by definition $\overline{f}(\bar{x}) := \inf_{z \in X} \left\{\varphi(\bar{x}z^{-1}) + F(z)\right\}$. By the density of X in $\overline{X}$ and the continuity of $\overline{\varphi}$ and F on $\overline{X}$ we have

$$\begin{aligned} \overline{f}(\bar{x}) &:= \inf_{z \in X} \left\{\varphi(\bar{x}z^{-1}) + F(z)\right\} \\ &= \inf_{z \in \overline{X}} \left\{\varphi(\bar{x}z^{-1}) + F(z)\right\} \\ &= \overline{\varphi} \oplus F \\ &= F \end{aligned}$$

The last equality follows from the fact that $F \in Lip^1_{0,\overline{\varphi}}(\overline{X}) = \mathcal{F}_{0,\overline{\varphi}}(\overline{X})$ by Lemma 3. Let us show now that χ is a monoid morphism. Indeed, let $f, g \in Lip^1_{0,\varphi}(X)$. Using the continuity of $\overline{f}$ and $\overline{g}$ and the density of X in $\overline{X}$, we easily see that $\overline{f} \oplus \overline{g}$ and $\overline{f \oplus g}$ coincide on X with $f \oplus g$, so by the injectivity of χ^{-1} we have $\overline{f} \oplus \overline{g} = \overline{f \oplus g}$. Thus χ is a monoid isomorphism. The fact that χ is isometric follow from the the density of X on $\overline{X}$ and a continuity argument.$\blacksquare$

For the proof of the following lemma see [[2], Lemma 1].

Lemma 5 *Let $f, g \in \mathcal{F}_0(X)$. Suppose that $d_\infty(f,g) := \sup_{x \in X} |f(x) - g(x)| < +\infty$ then*

$$d(f,g) := \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} = \frac{d_\infty(f,g)}{1 + d_\infty(f,g)}.$$

For each fixed point $x \in X$, the map δ^φ_x is defined on X by

$$\begin{aligned} \delta^\varphi_x : X &\rightarrow \mathbb{R} \\ z &\mapsto \varphi(zx^{-1}). \end{aligned}$$

We define the subset $\mathcal{G}^\varphi_0(X)$ of $\mathcal{F}_\varphi(X)$ by $\mathcal{G}^\varphi_0(X) := \{\delta^\varphi_x : x \in X\}$.

The following Lemma is an adaptation to our framework of [[2], Lemma 3].

Lemma 6 *Let $\varphi \in \mathcal{F}(X)$ be a remarkable idempotent. Then, the map*

$$\begin{aligned} \gamma^\varphi_X : (X, \Delta_{\infty,\varphi}) &\rightarrow (\mathcal{G}^\varphi_0(X), d_\infty) \\ x &\mapsto \delta^\varphi_x \end{aligned}$$

is a group isometric isomorphism. Or equivalently, the map

$$\begin{aligned} \gamma^\varphi_X : (X, \frac{\Delta_{\infty,\varphi}}{1 + \Delta_{\infty,\varphi}}) &\rightarrow (\mathcal{G}^\varphi_0(X), d) \\ x &\mapsto \delta^\varphi_x \end{aligned}$$

is a group isometric isomorphism.

Proof. The second part of the Lemma follow from the first part and Lemma 5, so we need to prove just the first part. Indeed, let $x_1, x_2 \in X$, we prove that $\delta^{\varphi}_{x_1} \oplus \delta^{\varphi}_{x_2} = \delta^{\varphi}_{x_1x_2}$. Indeed, let $x \in X$, we have :

$$\begin{array}{rcl} \delta^{\varphi}_{x_1} \oplus \delta^{\varphi}_{x_2}(x) &=& \inf_{z\in X} \left\{\delta^{\varphi}_{x_1}(xz^{-1}) + \delta^{\varphi}_{x_2}(z)\right\} \\ &\leq& \delta^{\varphi}_{x_1}(xx_2^{-1}) + \delta^{\varphi}_{x_2}(x_2) \\ &=& \varphi\left(x(x_1x_2)^{-1}\right) + \varphi(e_X) \\ &=& \delta^{\varphi}_{x_1x_2}(x). \end{array}$$

For the converse inequality, we use fact that $\varphi(xy) = \varphi(yx)$ for all $x, y \in X$ and the sub-additivity of φ , to obtain for all $z \in X$

$$\begin{array}{rcl} \delta^{\varphi}_{x_1x_2}(x) &=& \delta^{\varphi}_{x_1x_2}(x) \\ &=& \varphi\left(x(x_1x_2)^{-1}\right) \\ &=& \varphi(xx_2^{-1}x_1^{-1}) \\ &=& \varphi(x_1^{-1}(xx_2^{-1})) \\ &=& \varphi((x_1^{-1}xz^{-1})(zx_2^{-1})) \\ &\leq& \varphi(x_1^{-1}(xz^{-1})) + \varphi(zx_2^{-1}) \\ &=& \left(\varphi((xz^{-1})x_1^{-1})\right) + \varphi(zx_2^{-1}) \\ &=& \delta^{\varphi}_{x_1}(xz^{-1}) + \delta^{\varphi}_{x_2}(z) \end{array}$$

By taking the infinimum over z in the last inequality, we obtain

$$\delta^{\varphi}_{x_1x_2}(x) \leq \delta^{\varphi}_{x_1} \oplus \delta^{\varphi}_{x_2}(x).$$

Thus, $\delta^{\varphi}_{x_1} \oplus \delta^{\varphi}_{x_2} = \delta^{\varphi}_{x_1x_2}$. In other words,

$$\gamma^{\varphi}_X(x_1x_2) \quad = \quad \gamma^{\varphi}_X(x_1) \oplus \gamma^{\varphi}_X(x_2), \;\; \forall x_1, x_2 \in X. \tag{14}$$

Now by the definition of $\mathcal{G}^{\varphi}(X)$, γ^{φ}_X is a surjective map. Let us prove that γ^{φ}_X is one to one. Indeed, let $x_1, x_2 \in X$ be such that $\delta^{\varphi}_{x_1} = \delta^{\varphi}_{x_2}$ i.e $\varphi(xx_1^{-1}) = \varphi(xx_2^{-1})$ for all $x \in X$. Since φ satisfy the condition : $\varphi(x) = \varphi(x^{-1}) = 0 \Leftrightarrow x = e_X$ then by replacing x by x_1 in a first time and x by x_2 in a second time we obtain $0 = \varphi(e_X) = \varphi(x_1x_2^{-1}) = \varphi(x_2x_1^{-1}) = \varphi(\left(x_1x_2^{-1})^{-1}\right)$ which implies that $x_1x_2^{-1} = e_X$ i.e $x_1 = x_2$. Now, since X is a group and γ^{φ}_X is a bijective map satisfying the formula (14) then $(\mathcal{G}^{\varphi}_0(X), \oplus)$ is a group as image of group by an isomorphism. The identity element of $(\mathcal{G}^{\varphi}_0(X), \oplus)$ is of course $\gamma^{\varphi}_X(e_X) = \delta^{\varphi}_{e_X} = \varphi$. Thus γ^{φ}_X is a group isomorphism.

Let us show now that γ^{φ}_X is an isometry. By using the sub-additivity of φ and the fact that $\varphi(e_X) = 0$ we have :

$$\begin{array}{rcl} d_{\infty}(\delta^{\varphi}_{x_1}, \delta^{\varphi}_{x_2}) &=& \sup_{x\in X} |\delta^{\varphi}_{x_1}(x) - \delta^{\varphi}_{x_2}(x)| \\ &=& \sup_{x\in X} |\varphi(xx_1^{-1}) - \varphi(xx_2^{-1})| \\ &=& \max\left(\sup_{x\in X}(\varphi(xx_1^{-1}) - \varphi(xx_2^{-1})), \sup_{x\in X}(\varphi(xx_2^{-1}) - \varphi(xx_1^{-1}))\right) \\ &\leq& \max\left(\varphi(x_2x_1^{-1}), \varphi(x_1x_2^{-1})\right) \\ &=& \Delta_{\infty,\varphi}(x, y) \end{array}$$

For the inverse inequality,

$$
\begin{aligned}
d_\infty(\delta_{x_1}(\varphi), \delta_{x_2}(\varphi)) &= \sup_{x\in X} |\delta^\varphi_{x_1}(x) - \delta^\varphi_{x_2}(x)| \\
&= \sup_{x\in X} |\varphi(xx_1^{-1}) - \varphi(xx_2^{-1})| \\
&= \max\left(\sup_{x\in X}(\varphi(xx_1^{-1}) - \varphi(xx_2^{-1})), \sup_{x\in X}(\varphi(x-x_2) - \varphi(x-x_1))\right) \\
&\geq \max\left(\varphi(x_2x_1^{-1}); \varphi(x_1x_2^{-1})\right) \\
&= \Delta_{\infty,\varphi}(x,y)
\end{aligned}
$$

So $d_\infty(\delta^\varphi_{x_1}, \delta^\varphi_{x_2}) = \Delta_{\infty,\varphi}(x,y)$. This shows that γ^φ_X is an isometry. The second part of the Lemma follows from Lemma 5.$\blacksquare$

4.1.2 The semigroup $\mathcal{F}(X)$.

In this section we prove that using the following proposition, we can deduct results in the semigroup $\mathcal{F}(X)$ canonically from the results of the semi group $\mathcal{F}_0(X)$.
For all $(f,t),(g,s) \in \mathcal{F}_0(X)\times\mathbb{R}$, we denote by $(f,t)\overline{\oplus}(g,s) := (f\oplus g, t+s)$ and $d_1((f,t);(g,s)) := d(f,g) + |t-s|$. If φ is idempotent element, we denote by

$$\mathcal{F}_\varphi(X) := \{f \in \mathcal{F}(X) : f\oplus\varphi = \varphi\oplus f = f\}$$

the monoid having the identity element φ and by $Lip^1_\varphi(X)$ the following set

$$Lip^1_\varphi(X) := \left\{f \in \mathcal{F}(X) : f(x) - f(y) \leq \varphi(xy^{-1}); \forall x,y \in X\right\}.$$

If M is a monoid having φ as identity element, we denote by $\mathcal{U}(M)$ the group of unit of M i.e $\mathcal{U}(M) := \{f \in M/\exists g \in M : f\oplus g = g\oplus f = \varphi\}$.

Proposition 7 *Let X be a group. Then the following assertions hold,*

(1) *The following map is an isometric isomorphism of semigroups*

$$
\begin{aligned}
\pi : (\mathcal{F}(X), \oplus, d) &\to (\mathcal{F}_0(X)\times\mathbb{R}, \overline{\oplus}, d_1) \\
f &\mapsto (f - \inf_X f, \inf_X f).
\end{aligned}
$$

(2) *Let $\varphi \in \mathcal{F}(X)$ be a remarkable idempotent. Then*
(i) *$Lip^1_\varphi(X) = \mathcal{F}_\varphi(X)$ and $(Lip^1_\varphi(X), \oplus, d) \cong (Lip^1_{0,\varphi}(X)\times\mathbb{R}, \overline{\oplus}, d_1)$ as monoids.*
(ii) *$(\mathcal{U}(Lip^1_\varphi(X)), \oplus, d) \cong (\mathcal{U}(Lip^1_{0,\varphi}(X))\times\mathbb{R}, \overline{\oplus}, d_1)$ as groups.*

Proof. The part (1) is easy to verify. The part (2) follows from the fact that the isomorphism π send the monoid $\mathcal{F}_\varphi(X)$ on the monoid $\mathcal{F}_{0,\varphi}(X)\times\mathbb{R}$ and the fact that $\mathcal{F}_{0,\varphi}(X) = Lip^1_{0,\varphi}(X)$ by Lemma 3. Note also that $f \in Lip^1_\varphi(X)$ if and only if $f \inf_X f \in Lip^1_{0,\varphi}(X)$.$\blacksquare$

4.2 The main algebraic result: the group of unit.

Let us proof now our first main algebraic result announced in the introduction. **A. The monoid $Lip^1_{0,\varphi}(X)$.**

Theorem 6 *Let $\varphi \in \mathcal{F}(X)$ a remarkable idempotent. Then the following assertions hold.*

(1) $\mathcal{U}(Lip^1_{0,\varphi}(X)) = \chi^{-1} \circ \gamma^{\overline{\varphi}}_{\overline{X}}(\overline{X})$. *Where χ is the isometric isomorphism of Lemma 4 and $\gamma^{\overline{\varphi}}_{\overline{X}}$ is the isometric isomorphism of Lemma 6 applied to $(\overline{X}, \Delta_{\infty,\overline{\varphi}}) = (\overline{X}, \overline{\Delta}_{\infty,\varphi})$.*

(2) $(\mathcal{U}(Lip^1_{0,\varphi}(X)), d) \cong (\overline{X}, \frac{\overline{\Delta}_{\infty,\varphi}}{1+\overline{\Delta}_{\infty,\varphi}})$ *as groups.*

(3) $(\mathcal{U}(Lip^1_{0,\varphi}(X)), d_\infty) \cong (\overline{X}, \overline{\Delta}_{\infty,\varphi})$ *as groups.*

Proof. (1) By using Lemma 4 we have that $\chi(\mathcal{U}(Lip^1_{0,\varphi}(X))) = \mathcal{U}(Lip^1_{0,\overline{\varphi}}(\overline{X}))$ and by using Lemma 6 we have that $\gamma^{\overline{\varphi}}_{\overline{X}}(\overline{X}) = \mathcal{G}^{\overline{\varphi}}_0(\overline{X})$ so we need to prove that the group of unit $\mathcal{U}(Lip^1_{0,\overline{\varphi}}(\overline{X}))$ of $Lip^1_{0,\overline{\varphi}}(\overline{X})$ coincide with $\mathcal{G}^{\overline{\varphi}}_0(\overline{X})$.

$(*)$ $\mathcal{G}^{\overline{\varphi}}_0(\overline{X}) \subset \mathcal{U}(Lip^1_{0,\overline{\varphi}}(\overline{X}))$: this inclusion is clear since we now from Lemma 6 that $\mathcal{G}^{\overline{\varphi}}_0(\overline{X})$ is a group having $\overline{\varphi}$ as identity element.

$(**)$ $\mathcal{U}(Lip^1_{0,\overline{\varphi}}(\overline{X})) \subset \mathcal{G}^{\overline{\varphi}}_0(\overline{X})$: let $\overline{f} \in \mathcal{U}(Lip^1_{0,\overline{\varphi}}(\overline{X}))$, there exists $\overline{g} \in \mathcal{U}(Lip^1_{0,\overline{\varphi}}(\overline{X}))$ such that $\overline{f} \oplus \overline{g} = \overline{\varphi}$. Let us prove that the map $x \mapsto \overline{\varphi}(x) + \overline{\varphi}(x^{-1})$ has a stong minimum at e_X on $(\overline{X}, \Delta_{\infty,\overline{\varphi}}) = (\overline{X}, \overline{\Delta}_{\infty,\varphi})$. Indeed, since $\overline{\varphi}$ is remarkable idempotent then $\overline{\varphi} \geq 0 = \overline{\varphi}(e_X) = \overline{\varphi}(e_X^{-1})$ and so $x \mapsto \overline{\varphi}(x) + \overline{\varphi}(x^{-1})$ has a minimum at e_X. On the other hand $\Delta_{\infty,\overline{\varphi}}(x, e_X) = \max(\overline{\varphi}(x), \overline{\varphi}(x^{-1})) \leq \overline{\varphi}(x) + \overline{\varphi}(x^{-1})$. Now, $\overline{\varphi}(x_n) + \overline{\varphi}(x_n^{-1}) \to 0 \Rightarrow \Delta_{\infty,\overline{\varphi}}(x_n, e_X)$. Thus, the map $x \mapsto \overline{\varphi}(x) + \overline{\varphi}(x^{-1})$ has a strong minimum at e_X on the complet metric invariant group $(\overline{X}, \Delta_{\infty,\overline{\varphi}})$. Since $\overline{\varphi} = \overline{f} \oplus \overline{g}$ and since $\overline{f}$ and $\overline{g}$ are lower semi continuous (in fact 1-Lipschitz on $(\overline{X}, \Delta_{\infty,\overline{\varphi}})$), then we can apply Theorem 5 to obtain some $z_0 \in \overline{X}$ such that $\overline{f}(x) \geq \overline{\varphi}(xz_0) + \overline{f}(z_0^{-1})$ for all $x \in \overline{X}$. On the other hand since $\overline{f} \in Lip^1_{0,\overline{\varphi}}(\overline{X})$, then we have $\overline{f}(x) \leq \overline{\varphi}(xz_0) + \overline{f}(z_0^{-1})$ for all $x \in \overline{X}$. Thus $\overline{f}(x) = \overline{\varphi}(xz_0) + \overline{f}(z_0^{-1})$ for all $x \in \overline{X}$. Now since $\inf_{\overline{X}} \overline{f} = 0 = \inf_{\overline{X}} \overline{\varphi}$ then $\overline{f}(z_0^{-1}) = 0$. Finally, we have $\overline{f}(x) = \overline{\varphi}(xz_0) = \delta^{\overline{\varphi}}_{z_0{}^{-1}}(x)$ for all $x \in \overline{X}$ i.e $\overline{f} \in \mathcal{G}^{\overline{\varphi}}_0(\overline{X})$.

The part (2) and (3) are just interpretations of the part (1) with the fact that $d = \frac{d_\infty}{1+d_\infty}$ on $\mathcal{G}^{\overline{\varphi}}_0(\overline{X})$ by Lemma 5 since d_∞ is finite on this group by Lemma 6 .■

B. Abstract monoid $M_\varphi(X)$.

Definition 4 *Let S be a subset of , we say that S satisfy the translation property (T) if the following property hold :*
(T) The maps $x \mapsto f(zx)$ and $x \mapsto f(xz)$ belongs to $M_\varphi(X)$ for all $f \in M_\varphi(X)$ and all $z \in X$.

Proposition 8 *Let $\varphi \in \mathcal{F}(X)$ be a remarkable idempotent. Let $M_\varphi(X)$ be an abstract monoid of $\mathcal{F}_0(X)$ having φ as identity element. Then*

(1) *the group of unit $(\mathcal{U}(M_\varphi(X)), d)$ of $M_\varphi(X)$ is isometrically isomorphic to a subgroup G of $(\overline{X}, \frac{\overline{\Delta}_{\infty,\varphi}}{1+\overline{\Delta}_{\infty,\varphi}})$.*

(2) *If $M_\varphi(X)$ satisfy the property (T), then $(\mathcal{U}(M_\varphi(X)), d)$ is isometrically isomorphic to a subgroup of G such that $X \subset G \subset \overline{X}$.*

(3) *If the group X is complete for the metric $\Delta_{\infty,\varphi}$ and $M_\varphi(X)$ satisfy the property (T) then $\mathcal{U}(M_\varphi(X)), d) = (\mathcal{G}_0^\varphi(X), d)$ is isometrically isomorphic to $(X, \frac{\Delta_{\infty,\varphi}}{1+\Delta_{\infty,\varphi}})$.*

Proof. (1) We have $M_\varphi(X) \subset \mathcal{F}_{0,\varphi} = Lip_{0,\varphi}(X)$. So $\mathcal{U}(M_\varphi(X)) \subset \mathcal{U}(Lip_{0,\varphi}(X))$ which is isometrically isomorphic to $\overline{X}$ by Theorem 6. So the conclusion.

(2) If $M_\varphi(X)$ satisfy the property (T) then $\mathcal{G}_0^\varphi(X) \subset \mathcal{U}(M_\varphi(X))$ since $\mathcal{G}_0^\varphi(X)$ is a group included in $M_\varphi(X)$. On the other hand $\mathcal{G}_0^\varphi(X)$ is isometrically isomorphic to X by Lemma 6. This gives the conclusion with the part (1).

(3) The conclusion follow from the part (2) since $X = \overline{X}$ in this case.$\blacksquare$

Corollary 5 *Let (X, m) be complete metric invariant group. Let M be an abstract submonoid of the monoid $Lip_0^1(X)$ satisfying the translation property (T) Then the group of unit $(\mathcal{U}(M), d)$ is isometrically isomorphic to $(X, \frac{m}{1+m})$. This show that all submonoid M of $Lip_0^1(X)$ satisfing the property (T), have the same group of unit.*

Proof. The proof follow from the part (3) of Proposition 8 since in this case $\varphi = \varphi_m : x \mapsto m(x, e_X)$ and $(X, m) = (X, \Delta_{\infty,\varphi_m})$ is complete.$\blacksquare$

5 Applications to the Banach-Stone theorem.

Let us prove now our version of the Banach-Stone theorem which states that the structure of the monoid $(Lip_{0,\varphi}^1(X), \oplus, d)$ completely determine the structure of the metric invariant group completion $(\overline{X}, \overline{\Delta}_{\infty,\varphi})$ when φ is remarkable idempotent and symmetric.

Theorem 7 *Let X and Y be tow groups and let $\varphi \in \mathcal{F}(X)$ and $\psi \in \mathcal{F}(Y)$ be two remarkable idempotents. Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Rightarrow (6)$. If moreover we assume that φ and ψ are symetric (i.e $\varphi(x) = \varphi(x^{-1})$ and $\psi(y) = \psi(y^{-1})$ for all $x \in X$ and all $y \in Y$), then $(1) - (6)$ are equivalent.*

(1) *There exist a group isomorphism $T : \overline{X} \to \overline{Y}$ such that $\overline{\psi} \circ T = \overline{\varphi}$.*

(2) *There exist a semigroup isomorphism isometric $\Phi : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ sush that $\Phi(0) = 0$ and $\Phi(\overline{\varphi}) = \overline{\psi}$.*

(3) *There exist a semigroup isomorphism isometric $\Phi : \mathcal{F}_0(\overline{X}) \to \mathcal{F}_0(\overline{Y})$ sush that $\Phi(\overline{\varphi}) = \overline{\psi}$.*

(4) *$(Lip_{0,\overline{\varphi}}^1(\overline{X}), d) \cong (Lip_{0,\overline{\psi}}^1(\overline{Y}), d)$ as monoids.*

(5) *$(Lip_{0,\varphi}^1(X), d) \cong (Lip_{0,\psi}^1(Y), d)$ as monoids.*

(6) *$(\overline{X}, \overline{\Delta}_{\infty,\varphi}) \cong (\overline{Y}, \overline{\Delta}_{\infty,\psi})$ as groups.*

Proof. Note that we have $(\overline{X}, \overline{\Delta}_{\infty,\varphi}) = (\overline{X}, \Delta_{\infty,\overline{\varphi}})$.

$(1) \Rightarrow (2)$ If $T : \overline{X} \to \overline{Y}$ is an isomorphism such that $\overline{\psi} \circ T = \overline{\varphi}$ then the map $\Phi : \mathcal{F}(\overline{X}) \to \mathcal{F}(\overline{Y})$ defined by $\Phi(f) = f \circ T^{-1}$ is a semigroup isomorphism, isometric for the metric d and satisfy $\Phi(\overline{\varphi}) = \overline{\psi}$ and $\Phi(0) = 0$.

(2) $\Rightarrow$(3)$Since\Phi$ is a semigroup isomorphism that $\Phi(0 \oplus f) = \Phi(0) \oplus \Phi(f)$ for all $f \in \mathcal{F}(\overline{X})$. Since $\Phi(0) = 0$ and $0 \oplus f = \inf_X f$, then we obtain that $\Phi(\inf_X f) = \inf_Y \Phi(f)$ for all $\in \mathcal{F}(\overline{X})$. In particular, $0 = \Phi(0) = \inf_Y \Phi(f)$ for all $f \in \mathcal{F}_0(\overline{X})$, this show that Φ send $\mathcal{F}_0(\overline{X})$ on $\mathcal{F}_0(\overline{Y})$.

(3) $\Rightarrow$ (4) Since $\Phi(\overline{\varphi}) = \overline{\psi}$ and Φ is a semigroup isomorphism then clearly Φ maps the monoid $\mathcal{F}_{0,\overline{\varphi}}(\overline{X})$ onto the monoid $\mathcal{F}_{0,\overline{\psi}}(\overline{Y})$. So using the fact that $\mathcal{F}_{0,\overline{\varphi}}(\overline{X}) = Lip^1_{0,\overline{\varphi}}(\overline{X})$ and $\mathcal{F}_{0,\overline{\psi}}(\overline{Y}) = Lip^1_{0,\overline{\psi}}(\overline{Y})$ by Lemma 3, we obtain that $(Lip^1_{0,\overline{\varphi}}(\overline{X}), d) \cong (Lip^1_{0,\overline{\psi}}(\overline{Y}), d)$ as monoids by Φ.

(4) $\Leftrightarrow$ (5) Follows from Lemma 4.

(5) $\Rightarrow$ (6) Since $(Lip^1_{0,\varphi}(X), d) \cong (Lip^1_{0,\psi}(Y), d)$ and since isomorphism of monoids send the group of unit on the group of unit, we have $\mathcal{U}(Lip^1_{0,\varphi}(X)) \cong \mathcal{U}(Lip^1_{0,\varphi}(Y))$. Using Theorem 6 we obtain that $(\overline{X}, \overline{\Delta}_{\infty,\varphi}) \cong (\overline{Y}, \overline{\Delta}_{\infty,\psi})$.

Suppose now that φ and ψ are symmetric, then

$$\overline{\Delta}_{\infty,\varphi}(x, e_X) = \Delta_{\infty,\overline{\varphi}}(x, e_X) = \max(\overline{\varphi}(x), \overline{\varphi}(x^{-1}) = \overline{\varphi}(x)$$

for all $x \in \overline{X}$ and $\overline{\Delta}_{\infty,\psi}(y, e_Y) = \Delta_{\infty,\overline{\psi}}(y, e_Y) = \overline{\psi}(y)$ for all $y \in \overline{Y}$. We need to prove that (6) $\Rightarrow$ (1). Indeed, Let $T : (\overline{X}, \overline{\Delta}_{\infty,\varphi}) \to (\overline{Y}, \overline{\Delta}_{\infty,\psi})$ be an isomorphism isometric. In particular we have

$$\begin{aligned}
\overline{\psi}(T(x)) &= \Delta_{\infty,\overline{\psi}}(T(x), e_Y) \\
&= \Delta_{\infty,\overline{\psi}}(T(x), T(e_X)) \\
&= \Delta_{\infty,\overline{\varphi}}(x, e_X) \\
&= \overline{\varphi}(x).
\end{aligned}$$

This conclude the proof.■

The following corollary shows that the monoid $(Lip^1_0(X), \oplus, d)$ completely determine the structure of the complete metric invariant group (X, m).

Corollary 6 *Let (X, m) and (Y, m') be complete metric invariant groups. Let $\varphi_m : x \mapsto m(x, eX)$ for all $x \in X$ and $\psi_{m'} : y \mapsto m'(y, e_Y)$ for all $y \in Y$. Then, the following assertions are equivalent.*

(1) *$(X, m) \cong (Y, m')$ as groups.*

(2) *There exist a semigroup isomorphism isometric $\Phi : \mathcal{F}(X) \to \mathcal{F}(Y)$ sush that $\Phi(\varphi_m) = \psi_{m'}$ and $\Phi(0) = 0$.*

(3) *There exist a semigroup isomorphism isometric $\Phi : \mathcal{F}_0(X) \to \mathcal{F}_0(Y)$ sush that $\Phi(\varphi_m) = \psi_{m'}$.*

(4) *$(Lip^1_0(X), d) \cong (Lip^1_0(Y), d)$ as monoids.*

(5) *There exist a semigroup isomorphism isometric $\Phi : (Lip_0(X), d) \to (Lip_0(Y), d)$ sush that $\Phi(\varphi_m) = \psi_{m'}$.*

Proof. The part (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Theorem 7 and the fact that (X, m) and (Y, m') are complete, and that $Lip^1_0(X) = Lip^1_{0,\varphi_m}(X)$ with $\varphi_m : x \mapsto m(x, e_X)$.

On the other hand, the part (1) $\Rightarrow$ (5) is clear and the part (5) $\Rightarrow$ (4) follow from the fact that $Lip_0^1(X) = \mathcal{F}_{0,\varphi_m}(X)$, $Lip_0^1(X) = \mathcal{F}_{0,\varphi_{m'}}(Y)$ and the fact that Φ send necessarily the monoid $\mathcal{F}_{0,\varphi_m}(X)$ on the monoid $\mathcal{F}_{0,\varphi_{m'}}(Y)$.$\blacksquare$

Now we give the purely algebraic version of the above corollary. We denote by $\mathcal{F}_0^1(X)$ the monoid of all functions $f : X \to [0,1]$ such that $\inf_X f = 0$. This monoid has the identity element, the map

$$\delta_{e_X} = \begin{cases} 0 & if \;\; x = e_X \\ 1 & otherwise. \end{cases}$$

Corollary 7 *Let X and Y be two groups. Then the following assertions are equivalent.*

(1) *$(\mathcal{F}_0^1(X), \oplus, d)$ is isometrically isomorphic to $(\mathcal{F}_0^1(Y), \oplus, d)$ as monoids (also as semigroups).*

(2) *$(\mathcal{F}_0^1(X), \oplus)$ is isomorphic to $(\mathcal{F}_0^1(Y), \oplus)$ as monoids (also as semigroups).*

(3) *X and Y are isomorphic as groups.*

Proof. First, note that the group X (and in similar way the group Y) can be endowed with the discrete metric denoted by dis. So we have that (X, dis) is a complete metric invariant group. Then, we see easily that with this metric we have $\mathcal{F}_0^1(X) = Lip_0^1(X)$. On the other hand we have that X and Y are isomorphic if and only if (X, dis) and (Y, dis) are isometrically isomorphic, this implies by Corollary 6 that $(\mathcal{F}_0^1(X), d)$ and $(\mathcal{F}_0^1(Y), d)$ are isometrically isomorphic as monoids, in particular they are isomorphic. For the converse, if $\mathcal{F}_0^1(X)$ and $\mathcal{F}_0^1(Y)$ are isomorphic as monoids, then the group of unit of $\mathcal{F}_0^1(X)$ is isomorphic to the group of unit of $\mathcal{F}_0^1(Y)$. Thus, by Theorem 6 (Or Corollary 7) we obtain that X and Y are isomorphic.$\blacksquare$

6 Application to the Banach-Dieudonée Theorem.

Let us recall some notions. Let K and C be convex subsets of vector spaces. A function $T : K \to C$ is said to be affine if for all $x, y \in K$ and $0 \le \lambda \le 1$, $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$. The set of all continuous real-valued affine functions on a convex subset K of a topological vector space will be denoted by $Aff(K)$. We denote by $Aff_0(B_{X^*})$ the set of all affine weak star continuous functions that vanish at 0. Clearly, all translates of continuous linear functionals are elements of $Aff(K)$, but the converse in not true in general (see [9] page 21.). However, we do have the following relationship.

Proposition 9 *([9], Proposition 4.5) Assume that K is a compact convex subset of a separated locally convex space X then*

$$\left\{a \in Aff(K) : a = r + x^*_{|K} \text{ for some } x^* \in X^* \text{ and some } r \in R\right\}$$

is dense in $(Aff(K), \|.\|_\infty)$.

But in the particular case when X is a Banach space and $K = (B_{X^*}, w^*)$ is the unit ball of the dual space X^* endowed with the weak star topologies, the well known result due to Banach and Dieudonné states that :

$$Aff_0(B_{X^*}) = \left\{ \hat{z}_{|K} : z \in X \right\}.$$

Where $\hat{z} : p \mapsto p(z)$ for all $p \in X^*$ and $\hat{z}_{|B_{X^*}}$ denotes the restriction of $\hat{z}$ to K. In particular $(Aff_0(B_{X^*}), \|.\|_\infty)$ is isometrically isomorphic to $(X, \|.\|)$.

We give in this section a simple proof of the Banach-Dieudonné theorem by using our algebraic results of this article. More precisely, we us Theorem 6.

In what follow, K is a convex bounded and weak-star closed set of X^* congaing 0 such that $int(K) \neq \emptyset$ where $int(K)$ denote the interior of K for the norm topology. We denote by $\mathcal{A}(X^*)$ the set of all functions $F : X^* \to \mathbb{R} \cup \{+\infty\}$ convex weak-star lower semicontinuous with non empty domain. We denote by i_K the indicator function $i_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise and by $\mathcal{A}_K(X^*) := \{F + i_K : F \in \mathcal{A}(X^*)\}$ which is a monoid for the operation $+$ and having the function i_K as identity element. We also denote $\mathcal{A}(K)$ the set of all functions $F : K \to \mathbb{R} \cup \{+\infty\}$ convex weak-star lower semicontinuous on K, which is a monoid having 0 as identity element. Clearly, the map

$$\begin{array}{rcl} i : (\mathcal{A}(K), +) & \to & (\mathcal{A}_K(X^*), +) \\ F & \mapsto & \widetilde{F} + i_K. \end{array}$$

is a monoid isomorphism, where $\widetilde{F} := F$ on K and $+\infty$ otherwise.

Finally by $\mathcal{T}$ we denote the Fenchel-Moreau operator i.e $\mathcal{T}(f) = f^*$ where $f^*(p) := \sup_{x \in X} \{p(x) - f(x)\}$ for all $p \in X^*$ and by $\sigma_K : x \mapsto \sup_{p \in K} \{p(x)\}$ the support function. It is well known that the inf-convolution of two convex function is also convex function and that $(f \oplus g)^* = f^* + g^*$ is always true.

For the inf-convolution structure, we deal with the particular semigroup $\mathcal{CL}_0(X)$ of all convex map f defined on a Banach space X such that $\inf_X f = 0$ and the submonoid $M_{\sigma_K}(X) := \mathcal{CL}_0(X) \cap Lip_{0,\sigma_K}(X)$ of the monoid $(Lip_{0,\sigma_K}(X), \oplus)$ where $\sigma_K : x \mapsto \sup_{p \in K} \{p(x)\}$ denotes the support function which is a remarkable idempotent (see the proposition bellow). We recall below the well know Fenchel-Moreau theorem.

Theorem 8 (Fenchel-Moreau) *Let X be a Banach space and f be a function defined on X such that $\{x \in X : f(x) < +\infty\} \neq \emptyset$. Then, f is convexe lower semi continuous if and only if $f^{**}(x) = f(x)$ for all $x \in X$.*

Proposition 10 *Let K be a convex bounded and weak-star closed set of X^* such that $int(K) \neq \emptyset$. Then*

(1) *The support function σ_K is remarkable idempotent of $M_{\sigma_K}(X)$ and is Lipschitz on $(X, \|.\|)$.*

(2) *For each fixed point $z \in X$, we have $(\sigma_K(. - z))^* = \hat{z} + i_K$, where $\hat{z} : p \mapsto p(z)$ for all $p \in X^*$.*

(3) $\mathcal{U}(M_{\sigma_K}(X)) = \mathcal{G}_0^{\sigma_K}(X) := \{\sigma_K(. - z) : z \in X\}$.

(4) $\mathcal{U}(\mathcal{A}(K)) = Aff_0(K) = i^{-1}(\mathcal{U}(\mathcal{A}_K(X^*)))$

Proof. (1) We know that σ_K is subadditive, so from Proposition 6 it suffices to prove that $\sigma_K(x) = \sigma_K(-x) = 0$ if and only if $x = 0$ which is clear since $int(K) \neq \emptyset$. Since K is bounded set then σ_K is Lipschitz on $(X, \|.\|)$.

(2) This part is well known and can be easily verified.

(3) Since $M_{\sigma_K}(X)$ is stable by the translation property (T) (See Definition 4) then, by Proposition 8 we have $\mathcal{U}(M_{\sigma_K}(X)) = \mathcal{G}_0^{\sigma_K}(X) := \{\sigma_K(. - z) : z \in X\}$ since $(X, \Delta_{\infty,\sigma_K})$ is complete. In fact here $\Delta_{\infty,\sigma_K}(x, y) = \sup_{p\in K} |p(x - y)|$ is equivalent to the metric associated to the norm $\|.\|$ since K is bounded and $int(K) \neq \emptyset$.

(4) First we have $\mathcal{U}(\mathcal{A}(K)) = \mathcal{AFF}_0(K)$. Indeed, let $F \in \mathcal{U}(\mathcal{A}(K))$ then there exists $G \in \mathcal{U}(\mathcal{A}(K))$ such that $F + G = 0$ i.e $F = -G$ on K. Since both F and G are convex weak-star lower semicontinuous on K then F (respectivelly G) is affine and weak-star continuous on K. Conversely, if F is affine and weak-star continuous then $-F$ is affine and weak-star continuous too and so $F \in \mathcal{U}(\mathcal{A}(K))$. A monoid isomorphism send the group of unit on the group of unit so $i(\mathcal{U}(\mathcal{A}(K))) = i(Aff_0(K)) = \mathcal{U}(\mathcal{A}_K(X^*))$.■

Proposition 11 *Let K be a convex bounded and weak-star closed set of X^* such that $int(K) \neq \emptyset$.*

(2) *The map*

$$\begin{array}{rcl} \mathcal{T} : (M_{\sigma_K}(X), \oplus) & \to & (\mathcal{A}_K(X^*), +) \\ f & \mapsto & \mathcal{T}(f) \end{array}$$

is a monoid isomorphism. Consequently,

$$\begin{array}{rcl} i^{-1} \circ \mathcal{T} : (M_{\sigma_K}(X), \oplus) & \to & (\mathcal{A}(K), +) \\ f & \mapsto & i^{-1} \circ \mathcal{T}(f) \end{array}$$

is a monoid isomorphism.

Proof. The injectivity of $\mathcal{T}$ follow from the Fenchel-Moreau theorem (see above) since every element of $M_{\sigma_K}(X)$ is convex and Lipschitz. For the surjectivity, Let F be weak-star lower semicontinuous and convex function on X^*. Let $f : x \mapsto sup_{p\in K} \{p(x) - F(p)\} = (F + i_K)^*(x)$. Then $f \in M_{\sigma_K}(X)$ and $F + i_K = f^* = \mathcal{T}(f)$. Indeed, f is bounded from below since $0 \in K$ and $\inf_X f = 0$ since $F(0) = 0$. We have that f is convex and weak-star lower semicontinuous as supremum of affine and weak-star continuous functions. So $f \in \mathcal{CL}_0(X)$. On the other hand $f \in Lip_{0,\sigma_K}(X)$ as supremum of element in $Lip_{0,\sigma_K}(X)$, since for each fixed $p \in K$ we have $(p(x) - F(p)) - (p(y) - F(p)) = p(x - y) \leq \sigma_K(x - y)$ and we can take supremum in the inequality $(p(x) - F(p)) \leq \sigma_K(x - y) + (p(y) - F(p))$. Thus $f \in M_{\sigma_K}(X)$. Now by the Fenchel-Moreau theorem we have $F + i_K = (F + i_K)^{**} = f^*$ since $F + i_K$ is convex weak-star lower semicontinuous. So $\mathcal{T}$ is surjective. Finally $\mathcal{T}$ is a morphism for the inf-convolution since it is well know and easy to verify that $(f \oplus g)^* = f^* + g^*$ for all $f, g \in M_{\sigma_K}(X)$. By composition of isomorphism we obtain the second affirmation.■

We give now the algebraic proof of the Banach-Dieudonée theorem.

Theorem 9 (Banach-Dieudonée) *Let K be a convex bounded and weak-star closed set of X^* containing 0 such that $int(K) \neq \emptyset$. Then*

$$Aff_0(K) = \left\{ \hat{z}_{|K} : z \in X \right\}.$$

If moreover K is symmetric then $(Aff_0(K), \|.\|_\infty)$ is isometrically isomorphic to (X, σ_K). In this case σ_K is an equivalent norm on X.

Proof. Since a monoid isomorphism send the group of unit on the group of unit, then $i^{-1} \circ \mathcal{T}(\mathcal{U}(M_{\sigma_K}(X))) = \mathcal{U}(\mathcal{A}(K))$ by Proposition 11. The conclusion follow from Proposition 10.■

We give as corollary the following well known result.

Corollary 8 [[4], Theorem 55] *Let $F \in X^{**}$ (the bidual of X). Suppose that F is weak-star continuous. Then there exists $x \in X$ such that $F = \hat{x}$.*

Proof. The restriction $F_{|B_{X^*}}$ is affine weak-star continuous on B_{X^*}. So applying Theorem 9 with $K = B_{X^*}$, there exists $x \in X$ such that $F_{|B_{X^*}} = \hat{x}_{|B_{X^*}}$. By homogeneity we have $F = \hat{x}$.■

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