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Abstract. We give some reasonable and usable conditions on a sequence of norm one in a dual banach space under which the sequence does not converges to the origin in the w^*-topology. These requirements help to ensure that the Lagrange multipliers are nontrivial, when we are interested for example on the infinite dimensional infinite-horizon Pontryagin Principles for discrete-time problems.

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1 Introduction.

Let Z be a Banach space and Z^* its topological dual. It is well known that in infinite dimensional separable Banach space, it is always true that the origin in Z^* is the w^*-limit of a sequence from the unit sphere S_{Z^*} as it is in its w^*-closure. In this paper, we look about reasonable and usable conditions on a sequence of norm one in Z^* such that this sequence does not converge to the origin in the w^*-topology. This situation has the interest, when we are looking for a nontrivial Lagrange multiplier for optimization problems, and was encountered several times in the literature. See for example [1] and [3]. To guarantee that the multiplier are nontrivial at the limit, the authors in [3] used the following lemma from [[2], pp. 142, 135].

Definition 1. *A subset Q of a Banach space Z is said to be of finite codimension in Z if there exists a point z_0 in the closed convex hull of Q such that the closed vector space generated by $Q - z_0 := \{q - z_0 |\ q \in Q\}$ is of finite codimension in Z and the closed convex hull of $Q - z_0$ has a no empty interior in this vector space.*

Lemma 1. *([2], pp. 142, 135) Let $Q \subset Z$ be a subset of finite codimension in Z. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that*

i) $\|f_k\| \geq \delta > 0$, *for all* $k \in \mathbb{N}$ *and* $f_k \xrightarrow{w^*} f$.

ii) *for all* $z \in Q$, *and for all* $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k$.

Then, $f \neq 0$.

Note that, this is not the most general situation. Indeed, one can meet as in [1], a situation where the part *ii)* of the above lemma is not uniform on $z \in Z$, and depends on other parameter as follows: for all $z \in \overline{co}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$. The principal Lemma 4 that we propose in this paper, will permit to include this very useful situation. This lemma is based on the Baire category theorem.

2 Preliminary Lemmas.

We need the following classical lemma. We denote by $Int(A)$ the topological interior of a set A.

Lemma 2. *Let C be a convex subset of a normed vector space. Let $x_0 \in Int(C)$ and $x_1 \in \overline{C}$. Then, for all $\alpha \in]0, 1]$, we have $\alpha x_0 + (1-\alpha)x_1 \in Int(C)$.*

We deduce the following lemma.

Lemma 3. *Let $(F, \|.\|_F)$ be a Banach space and C be a closed convex subset of F with non empty interior. Suppose that $D \subset C$ is a closed subset of C with no empty interior in $(C, \|.\|_F)$ (for the topology induced by C). Then, the interior of D is non empty in $(F, \|.\|_F)$.*

Proof. On one hand, there exists x_0 such that $x_0 \in Int(C)$. On the other hand, since D has no empty interior in $(C, \|.\|_F)$, there exists $x_1 \in D$ and $\epsilon_1 > 0$ such that $B_F(x_1, \epsilon_1) \cap C \subset D$. By using Lemma 2, $\forall \alpha \in]0, 1]$, we have $\alpha x_0 + (1-\alpha)x_1 \in Int(C)$. Since $\alpha x_0 + (1-\alpha)x_1 \to x_1$ when $\alpha \to 0$, then there exist some small α_0 and an integer number $N \in \mathbb{N}^*$ such that $B_F(\alpha_0 x_0 + (1-\alpha_0)x_1, \frac{1}{N}) \subset B(x_1, \epsilon_1) \cap C \subset D$. Thus D has a non empty interior in F. $\square$

3 The principal Lemma.

We give now our principlal lemma. We denote by $\overline{co}(X)$ the closed convex hull of X.

Lemma 4. *Let Z be a Banach space. Let $(p_n)_n$ be a sequence of subadditive and continuous map on Z and $(\lambda_n)_n \subset \mathbb{R}^+$ be a sequence of nonegative real number such that $\lambda_n \to 0$. Let A be a non empty subset of Z, $a \in \overline{co}(A)$ and $F := \overline{span(A-a)}$ the closed vector space generated by A. Suppose that $\overline{co}(A-a)$ has no empty interior in F and that*

(1) *for all* $z \in \overline{co}(A)$, *there exists* $C_z \in \mathbb{R}$ *such that for all* $n \in \mathbb{N}$:

$$p_n(z) \leq C_z \lambda_n.$$

(2) *for all* $z \in F$, $\limsup_n p_n(z) \leq 0$.

Then, for all bounded subset B *of* F, *we have*

$$\limsup_n \left(\sup_{h \in B} p_n(h) \right) \quad \leq \quad 0.$$

Proof. For each $m \in \mathbb{N}$, we set

$$F_m := \{ z \in \overline{co}(A) : p_n(z) \leq m\lambda_n, \forall n \in \mathbb{N} \}.$$

The sets F_m are closed subsets of Z. Indeed,

$$F_m = \left(\bigcap_{n \in \mathbb{N}} p_n^{-1}(] - \infty, m\lambda_n]) \right) \cap (\overline{co}(A))$$

where, for each $n \in \mathbb{N}$, $p_n^{-1}(] - \infty, m\lambda_n])$ is a closed subset of Z by the continuity of p_n. On the other hand, we have $\overline{co}(A) = \bigcup_{m \in N} F_m$. Indeed, let $z \in \overline{co}(A)$, there exists $C_z \in \mathbb{R}$ such that $p_n(z) \leq C_z \lambda_n$ for all $n \in \mathbb{N}$. If $C_z \leq 0$, then $z \in F_0$. If $C_z > 0$, it suffices to take $m_1 := [C_z] + 1$ where $[C_z]$ denotes the floor of C_z to have that $z \in F_{m_1}$. We deduce then that $F_m - a$ are closed and that $\overline{co}(A) - a = \bigcup_{m \in N} (F_m - a)$. Using the Baire Theorem on the complete metric space $(\overline{co}(A) - a, \|.\|_F)$, we get an $m_0 \in \mathbb{N}$ such that $F_{m_0} - a$ has no empty interior in $(\overline{co}(A) - a, \|.\|_F)$. Since by hypothesis $\overline{co}(A) - a$ has no empty interior in F, using Lemma 3 to obtain that $F_{m_0} - a$ has no empty interior in F. So there exists $z_0 \in F_{m_0} - a$ and some integer number $N \in \mathbb{N}^*$ such that $B_F(z_0, \frac{1}{N}) \subset F_{m_0} - a$. In other words, for all $z \in B_F(b, \frac{1}{N}) \subset F_{m_0}$ (with $b := a + z_0 \in F_{m_0} \subset F$) and all $n \in \mathbb{N}$, we have:

$$p_n(z) \leq m_0 \lambda_n. \tag{1}$$

Let now B a bounded subset of F, there exists an integer number $K_B \in \mathbb{N}^*$ such that $B \subset B_F(0, K_B)$. On the other hand, for all $h \in B$, there exists $z_h \in B_F(b, \frac{1}{N})$ such that $h = K_B.N(z_h - b)$. So using (1) and the subadditivity of p_n, we obtain that, for all $n \in \mathbb{N}$:

$$\begin{aligned} p_n(h) = p_n(K_B N(z_h - b)) &\leq K_B N p_n(z_h - b) \\ &\leq K_B N (p_n(z_h) + p_n(-b)) \\ &\leq K_B N m_0 \lambda_n + K_B N p_n(-b). \end{aligned}$$

On passing to the supremum on B, we obtain for all $n \in \mathbb{N}$,

$$\sup_{h \in B} p_n(h) \leq K_B N m_0 \lambda_n + K_B N p_n(-b).$$

Since $\lambda_n \longrightarrow 0$, we have

$$\limsup_n \left(\sup_{h \in B} p_n(h) \right) \quad \leq \quad K_B N \limsup_n p_n(-b) \leq 0.$$

This concludes the proof. $\square$

As an immediat consequence, we obtain the following corollary.

Corollary 1. *Let Z be a Banach space. Let $(f_n)_n \subset Z^*$ be a sequence of linear and continuous functionnals on Z and let $(\lambda_n)_n \subset \mathbb{R}^+$ such that $\lambda_n \to 0$. Let A be a no empty subset of Z, $a \in \overline{co}(A)$ and $F := \overline{span(A-a)}$ the closed vector space generated by A. Suppose that $\overline{co}(A-a)$ $(= \overline{co}(A) - a)$ has no empty interior in F and that*

(1) *for all $z \in \overline{co}(A)$, there exists a real number C_z such that, for all $n \in \mathbb{N}$, we have*

$$f_n(z) \leq C_z \lambda_n.$$

(2) $f_n \overset{w^*}{\to} 0$.

Then, $\|(f_n)_{|F}\|_{F^*} \to 0$.

Proof. The proof follows Lemma 4 with the subadditive and continuous maps f_n and the bounded set $B := S_{F^*}$. □

In the following corollary, the inequality in $ii)$ depends on $z \in Z$ unlike in [2] where the inequality is uniformly independent on z. Note also that if C_z does not depend on z, the condition $ii)$ is also true by replacing: for all $z \in \overline{co}(Q)$ by for all $z \in Q$.

Corollary 2. *Let $Q \subset Z$ be a subset of finite codimension in Z. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that*

i) $\|f_k\| \geq \delta > 0$, *for all $k \in \mathbb{N}$, and* $f_k \overset{w^*}{\to} f$.

ii) for all $z \in \overline{co}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$.

Then, $f \neq 0$.

Proof. Suppose by contradiction that $f = 0$. By applying Corollary 1 to Q and $-f_k$, we obtain $\|(f_k)_{|F}\|_{F^*} \to 0$ where $F := \overline{span(Q - z_0)}$. Since F is of finite codimension in Z, there exists a finite-dimensional subspace E of Z, such that $Z = F \oplus E$. Thus, there exists $L > 0$ such that

$$\|f_k\|_Z \leq L \left(\|(f_k)_{|E}\|_{E^*} + \|(f_k)_{|F}\|_{F^*} \right).$$

Then, using $i)$ we obtain $\lim_k \|(f_k)_{|E}\|_{E^*} \geq \frac{\delta}{L}$. Since the weak-star topology and the norm topology coincids on E because of finite dimension, we have that $0 = \|(f)_{|E}\|_{E^*} = \lim_k \|(f_k)_{|E}\|_{E^*} \geq \frac{\delta}{L} > 0$, which is a contradiction. Hence $f \neq 0$. □

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