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Abstract. We give some reasonable and usable conditions on a sequence of norm one in a dual banch space under which the sequence does not converges to the origin in the $w^*$-topology. These requirements help to ensure that the Lagrange multipliers are nontrivial, when we are interested for example on the infinite dimensional infinite-horizon Pontryagin Principles for discrete-time problems.

Keyw ord, phrase: Baire category theorem, Subadditive and continuous map, Multiplier rules.

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1 Introduction.

Let $Z$ be a Banach space and $Z^*$ its topological dual. It is well known that in infinite dimensional separable Banach space, it is always true that the origin in $Z^*$ is the $w^*$-limit of a sequence from the unit sphere $S_{Z^*}$ as it is in its $w^*$-closure. In this paper, we look about reasonable and usable conditions on a sequence of norm one in $Z^*$ such that this sequence does not converges to the origin in the $w^*$-topology. This situation has the interest, when we are looking for a nontrivial Lagrange multiplier for optimization problems, and was encountered several times in the literature. See for example [1] and [3]. To guarantee that the multiplier are nontrivial at the limit, the authors in [3] used the following lemma from [[2], pp. 142, 135].

Definition 1. A subset $Q$ of a Banach space $Z$ is said to be of finite codimension in $Z$ if there exists a point $z_0$ in the closed convex hull of $Q$ such that the closed vector space generated by $Q - z_0 := \{q - z_0 | q \in Q\}$ is of finite codimension in $Z$ and the closed convex hull of $Q - z_0$ has a no empty interior in this vector space.
Lemma 1. ([2], pp. 142, 135) Let $Q \subset Z$ be a subset of finite codimension in $Z$. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that

i) $\|f_k\| \geq \delta > 0$, for all $k \in \mathbb{N}$ and $f_k \overset{w^*}{\to} f$.

ii) for all $z \in Q$, and for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k$.

Then, $f \neq 0$.

Note that, this is not the most general situation. Indeed, one can meet as in [1], a situation where the part ii) of the above lemma is not uniform on $z \in Z$, and depends on other parameter as follows: for all $z \in \text{co}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$. The principal Lemma 4 that we propose in this paper, will permit to include this very useful situation. This lemma is based on the Baire category theorem.

2 Preliminary Lemmas.

We need the following classical lemma. We denote by $\text{Int}(A)$ the topological interior of a set $A$.

Lemma 2. Let $C$ be a convex subset of a normed vector space. Let $x_0 \in \text{Int}(C)$ and $x_1 \in C$. Then, for all $\alpha \in [0, 1]$, we have $\alpha x_0 + (1-\alpha)x_1 \in \text{Int}(C)$.

We deduce the following lemma.

Lemma 3. Let $(F, \|\|_F)$ be a Banach space and $C$ be a closed convex subset of $F$ with non empty interior. Suppose that $D \subset C$ is a closed subset of $C$ with no empty interior in $(C, \|\|_F)$ (for the topology induced by $C$). Then, the interior of $D$ is non empty in $(F, \|\|_F)$.

Proof. On one hand, there exists $x_0$ such that $x_0 \in \text{Int}(C)$. On the other hand, since $D$ has no empty interior in $(C, \|\|_F)$, there exists $x_1 \in D$ and $\epsilon_1 > 0$ such that $B_F(x_1, \epsilon_1) \cap C \subset D$. By using Lemma 2, $\forall \alpha \in [0, 1]$, we have $\alpha x_0 + (1-\alpha)x_1 \in \text{Int}(C)$. Since $\alpha x_0 + (1-\alpha)x_1 \to x_1$ when $\alpha \to 0$, then there exist some small $\alpha_0$ and an integer number $N \in \mathbb{N}^*$ such that $B_F(\alpha x_0 + (1-\alpha_0)x_1, \frac{1}{N}) \subset B(x_1, \epsilon_1) \cap C \subset D$. Thus $D$ has a non empty interior in $F$. \hfill $\square$

3 The principal Lemma.

We give now our principal lemma. We denote by $\text{co}(X)$ the closed convex hull of $X$.

Lemma 4. Let $Z$ be a Banach space. Let $(p_n)_n$ be a sequence of subadditive and continuous map on $Z$ and $(\lambda_n)_n \subset \mathbb{R}^+$ be a sequence of nonnegative real number such that $\lambda_n \to 0$. Let $A$ be a non empty subset of $Z$, $a \in \text{co}(A)$ and $F := \text{span}(A - a)$ the closed vector space generated by $A$. Suppose that $\text{co}(A - a)$ has no empty interior in $F$ and that

1. for all $z \in \text{co}(A)$, there exists $C_z \in \mathbb{R}$ such that for all $n \in \mathbb{N}$:

$$p_n(z) \leq C_z \lambda_n.$$
(2) for all \( z \in F \), \( \limsup_n p_n(z) \leq 0 \).

Then, for all bounded subset \( B \) of \( F \), we have
\[
\limsup_n \left( \sup_{h \in B} p_n(h) \right) \leq 0.
\]

**Proof.** For each \( m \in \mathbb{N} \), we set
\[
F_m := \{ z \in \overline{\mathcal{O}}(A) : p_n(z) \leq m\lambda_n, \forall n \in \mathbb{N} \}.
\]

The sets \( F_m \) are closed subsets of \( Z \). Indeed,
\[
F_m = \left( \bigcap_{n \in \mathbb{N}} p_n^{-1}([0, m\lambda_n]) \right) \cap (\overline{\mathcal{O}}(A))
\]
where, for each \( n \in \mathbb{N} \), \( p_n^{-1}([0, m\lambda_n]) \) is a closed subset of \( Z \) by the continuity of \( p_n \). On the other hand, we have \( \overline{\mathcal{O}}(A) = \bigcup_{m \in \mathbb{N}} F_m \). Indeed, let \( z \in \overline{\mathcal{O}}(A) \), there exists \( C_z \in \mathbb{R} \) such that \( p_n(z) \leq C_z\lambda_n \) for all \( n \in \mathbb{N} \). If \( C_z \leq 0 \), then \( z \in F_0 \). If \( C_z > 0 \), it suffices to take \( m_1 := \lfloor C_z \rfloor + 1 \) where \( \lfloor C_z \rfloor \) denotes the floor of \( C_z \) to have that \( z \in F_{m_1} \). We deduce then that \( F_m - a \) are closed and that \( \overline{\mathcal{O}}(A) - a = \bigcup_{m \in \mathbb{N}} (F_m - a) \). Using the Baire Theorem on the complete metric space \( (\overline{\mathcal{O}}(A) - a, \| \|_F^d) \), we get an \( m_0 \in \mathbb{N} \) such that \( F_{m_0} - a \) has no empty interior in \( (\overline{\mathcal{O}}(A) - a, \| \|_F^d) \). Since by hypothesis \( \overline{\mathcal{O}}(A) - a \) has no empty interior in \( F \), using Lemma 3 to obtain that \( F_{m_0} - a \) has no empty interior in \( F \). So there exists \( z_0 \in F_{m_0} - a \) and some integer number \( N \in \mathbb{N}^* \) such that \( B_F(z_0, \frac{1}{N}) \subset F_{m_0} - a \). In other words, for all \( z \in B_F(b, \frac{1}{N}) \subset F_{m_0} \) (with \( b := a + z_0 \in F_{m_0} \subset F \)) and all \( n \in \mathbb{N} \), we have:
\[
p_n(z) \leq m_0\lambda_n. \tag{1}
\]

Let now \( B \) a bounded subset of \( F \), there exists an integer number \( K_B \in \mathbb{N}^* \) such that \( B \subset B_F(0, K_B) \). On the other hand, for all \( h \in B \), there exists \( z_h \in B_F(b, \frac{1}{N}) \) such that \( h = K_B N(z_h - b) \). So using (1) and the subadditivity of \( p_n \), we obtain that, for all \( n \in \mathbb{N} \):
\[
p_n(h) = p_n(K_B N(z_h - b)) \leq K_B N p_n(z_h - b) \\
\leq K_B N (p_n(z_h) + p_n(-b)) \\
\leq K_B N m_0 \lambda_n + K_B N p_n(-b).
\]

On passing to the supremum on \( B \), we obtain for all \( n \in \mathbb{N} \),
\[
\sup_{h \in B} p_n(h) \leq K_B N m_0 \lambda_n + K_B N p_n(-b).
\]

Since \( \lambda_n \to 0 \), we have
\[
\limsup_n \left( \sup_{h \in B} p_n(h) \right) \leq K_B N \limsup_n p_n(-b) \leq 0.
\]

This concludes the proof.
As an immediate consequence, we obtain the following corollary.

**Corollary 1.** Let $Z$ be a Banach space. Let $(f_n)_n \subset Z^*$ be a sequence of linear and continuous functionals on $Z$ and let $(\lambda_n)_n \subset \mathbb{R}^+$ such that $\lambda_n \to 0$. Let $A$ be a non-empty subset of $Z$, $a \in \overline{\mathcal{O}}(A)$ and $F := \text{span}(A - a)$ the closed vector space generated by $A$. Suppose that $\overline{\mathcal{O}}(A - a) (= \overline{\mathcal{O}}(A) - a)$ has no empty interior in $F$ and that

1. for all $z \in \overline{\mathcal{O}}(A)$, there exists a real number $C_z$ such that, for all $n \in \mathbb{N}$, we have $f_n(z) \leq C_z \lambda_n$.

2. $f_n \overset{w^*}{\to} 0$.

Then, $\|(f_n)|_F\|_{F^*} \to 0$.

**Proof.** The proof follows Lemma 4 with the subadditive and continuous maps $f_n$ and the bounded set $B := S_{F^*}$. 

In the following corollary, the inequality in $ii)$ depends on $z \in Z$ unlike in [2] where the inequality is uniformly independent on $z$. Note also that if $C_z$ does not depend on $z$, the condition $ii)$ is also true by replacing: for all $z \in \overline{\mathcal{O}}(Q)$ by for all $z \in Q$.

**Corollary 2.** Let $Q \subset Z$ be a subset of finite codimension in $Z$. Let $(f_k)_k \subset Z^*$ and $\epsilon_k \geq 0$ and $\epsilon_k \to 0$ such that

1. $\|f_k\| \geq \delta > 0$, for all $k \in \mathbb{N}$, and $f_k \overset{w^*}{\to} f$.

2. for all $z \in \overline{\mathcal{O}}(Q)$, there exists $C_z \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, $f_k(z) \geq -\epsilon_k C_z$.

Then, $f \neq 0$.

**Proof.** Suppose by contradiction that $f = 0$. By applying Corollary 1 to $Q$ and $-f_k$, we obtain $\|(f_k)|_F\|_{F^*} \to 0$ where $F := \text{span}(Q - z_0)$. Since $F$ is of finite codimension in $Z$, there exists a finite-dimensional subspace $E$ of $Z$, such that $Z = F \oplus E$. Thus, there exists $L > 0$ such that

$$\|f_k\|_Z \leq L \left(\|(f_k)|_E\|_{E^*} + \|(f_k)|_F\|_{F^*}\right).$$

Then, using $i)$ we obtain $\lim_k \|(f_k)|_E\|_{E^*} \geq \frac{\delta}{L}$. Since the weak-star topology and the norm topology coincides on $E$ because of finite dimension, we have that $0 = \|(f)|_E\|_{E^*} = \lim_k \|(f_k)|_E\|_{E^*} \geq \frac{\delta}{L} > 0$, which is a contradiction. Hence $f \neq 0$. 

**References**

