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February 12, 2016

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Abstract. We characterize the limited operators by differentiability of convex continuous functions. Given Banach spaces Y and X and a linear continuous operator $T : Y \longrightarrow X$, we prove that T is a limited operator if and only if, for every convex continuous function $f : X \longrightarrow \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever f is Gâteaux differentiable at $T(y) \in X$.

Keyword, phrase: Limited operators, Gâteaux differentiability, Fréchet differentiability, convex functions, extreme points.
2010 Mathematics Subject: Primary 46G05, 49J50, 58C20, 46B20, secondary 47B07.

1 Introduction.

A subset A of a Banach space X is called limited, if every weak* null sequence $(p_n)_n$ in X^* converges uniformly on A, that is,

$$\lim_{n\to+\infty} \sup_{x\in A} |\langle p_n, x\rangle| = 0.$$

We know that every relatively compact subset of X is limited, but the converse is false in general. A bounded linear operator $T : Y \longrightarrow X$ between Banach spaces Y and X is called limited, if T takes the closed unit ball B_Y of Y to a limited subset of X. It is easy to see that $T : Y \longrightarrow X$ is limited if and only if, the adjoint operator $T^* : X^* \longrightarrow Y^*$ takes weak* null sequence to norm null sequence. For useful properties of limited sets and limited operators we refer to [11], [4], [6] and [1].

We know that in a finite dimensional Banach space, the notions of Gâteaux and Fréchet differentiability coincide for convex continuous functions. In [5], Borwein and Fabian proved that a Banach space Y is infinite dimensional if and only if, there exists on Y functions f having points at which f is Gâteaux but not Fréchet differentiable. They also pointed in the introduction of [5] the observation that if the sup-norm $\|.\|_\infty$ on c_0 is Gâteaux differentiable at some point, then it is Fréchet differentiable there. In this article we observe that this phenomenon is not just related to the sup-norm but more generally, for each convex lower semicontinuous function $g : l^\infty \to \mathbb{R}\cup\{+\infty\}$, if g is Gâteaux differentiable at some point $a \in c_0$ which is in the interior of its domain, then the restriction of g to c_0 is Fréchet differentiable at a. This hold in particular when $g = (f^*)^*$ is the Fenchel biconjugate of a convex continuous function $f : c_0 \to \mathbb{R}$. In fact, this phenomenon is due, (see Corollary 1 in the Appendix and the comment just before), to the fact that the canonical embedding $i : c_0 \longrightarrow l^\infty$ is a limited operator (see the reference [6]).

The goal of this paper, is to prove the following characterization of limited operators in terms of the coincidence of Gâteaux and Fréchet differentiability of convex continuous functions.

Theorem 1. *Let Y and X be two Banach spaces and $T : Y \longrightarrow X$ be a continuous linear operator. Then, T is a limited operator if and only if, for every convex continuous function $f : X \longrightarrow \mathbb{R}$ and every $y \in Y$, the function $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever f is Gâteaux differentiable at $T(y) \in X$.*

As consequence we give, in Theorem 2 below, new characterizations of infinite dimensional Banach spaces, complementing a result of Borwein and Fabian in [[5], Theorem 1.].

A real valued function f on a Banach space will be called a PGNF-function (see [5]) if there exists a point at which f is Gâteaux but not Fréchet differentiable. A JN-sequence (due to Josefson-Nissenzweig theorem, see [[7], Chapter XII]) is a sequence $(p_n)_n$ in a dual space Y^* that is weak* null and $\inf_n \|p_n\| > 0$. We say that a function g on X^* has a norm-strong minimum (resp. weak*-strong minimum) at $p \in X^*$ if $g(p) = \inf_{q \in X^*} g(q)$ and $(p_n)_n$ norm converges (resp. weak* converges) to p whenever $g(p_n) \longrightarrow g(p)$. A norm-strong minimum and weak*-strong minimum are in particular unique.

Theorem 2. *Let Y be a Banach space. Then the following assertions are equivalent.*

(1) *Y is infinite dimensional.*

(2) *There exists a JN-sequence in Y^*.*

(3) *There exists a convex norm separable and weak* compact metrizable subset K of Y^* contaning 0 and a continuous seminorm h on X^* which is weak* lower semicontinuous and weak* sequentially continuous, such that the restriction $h_{|K}$ has a weak*-strong minimum but not norm-strong minimum at 0.*

(4) *There exists a Banach space X and a linear continuous non-limited operator $T : Y \longrightarrow X$.*

(5) *There exists on Y a convex continuous PGNF-function.*

In Section 2 we give some preliminary results, specially the key Lemma 2. In Section 3, we give the proof of Theorem 1 (divided into two part, Theorem 3 and Theorem 4) and the proof of Theorem 2. In Section 4 we give some complementary remarks.

2 Preliminaries.

We recall the following classical result.

Lemma 1. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set K such that*

(1) *K is Hausdorff with respect $\mathcal{T}_1$,*

(2) *K is compact with respect to $\mathcal{T}_2$,*

(3) *$\mathcal{T}_1 \subset \mathcal{T}_2$.*

Then $\mathcal{T}_1 = \mathcal{T}_2$.

Proof. Let $F \subset K$ be a $\mathcal{T}_2$-closed set. It follows that F is $\mathcal{T}_2$-compact, since K is $\mathcal{T}_2$-compact. Let $\{\mathcal{O}_i : i \in I\}$ be any cover of F by $\mathcal{T}_1$-open sets. Since $\mathcal{T}_1 \subset \mathcal{T}_2$, then each of these sets is also $\mathcal{T}_2$-open. Hence, there exist a finite subcollection that covers F. It follows that F is $\mathcal{T}_1$-compact and therefore is $\mathcal{T}_1$-closed since $\mathcal{T}_1$ is Hausdorff. This implies that $\mathcal{T}_2 \subset \mathcal{T}_1$. Consequently, $\mathcal{T}_1 = \mathcal{T}_2$.

□

Now, we establish the following useful lemma. If B is a subset of a dual Banach space X^*, we denote by $\overline{co}^{w^*}(B)$ the weak* closed convex hull of B.

Lemma 2. *Let X be a Banach space and K be a subset of X^*.*

(1) *Suppose that K is norm separable, then there exists a sequence $(x_n)_n$ in the unit sphere S_X of X which separate the points of K i.e. for all $p, p' \in K$, if $\langle p, x_n\rangle = \langle p', x_n\rangle$ for all $n \in \mathbb{N}$, then $p = p'$. Consequently, if K is a weak* compact and norm separable set of X^*, then the weak* topology of X^* restricted to K is metrizable.*

(2) *Let $(p_n)_n$ be a weak* null sequence in X^*. Then, the set $\overline{co}^{w^*}\{p_n : n \in \mathbb{N}\}$ is convex weak* compact and norm separable.*

Proof. (1) Since K is norm separable, then $K - K := \{a - b/(a, b) \in K \times K\}$ is also norm separable and so there exists a sequence $(q_n)_n$ of $K - K$ which is dense in $K - K$. According to the Bishop-Phelps theorem [3], the set

$$D = \{r \in X^* \mid r \text{ attains its supremum on the sphere } S_X\}$$

is norm-dense in the dual X^*. Thus, for each $n \in \mathbb{N}$, there exists $r_n \in D$ such that $\|q_n - r_n\| < \frac{1}{1+n}$. For each $n \in \mathbb{N}$, let $x_n \in S_X$ be such that $\|r_n\| = \langle r_n, x_n\rangle$. We claim that the sequence $(x_n)_n$ separate the points of K. Indeed, let $q \in K - K$ and suppose that $\langle q, x_n\rangle = 0$, for all $n \in \mathbb{N}$. There exists a subsequence $(q_{n_k})_k \subset K - K$ such that $\|q_{n_k} - q\| < \frac{1}{k}$ for all $k \in \mathbb{N}^*$ and so we have $\|r_{n_k} - q\| < \frac{1}{1+n_k} + \frac{1}{k}$. It follows that

$$\begin{aligned} \|r_{n_k}\| &= \langle r_{n_k}, x_{n_k}\rangle \\ &= \langle r_{n_k}, x_{n_k}\rangle - \langle q, x_{n_k}\rangle \\ &\leq \|r_{n_k} - q\| \\ &< \frac{1}{1+n_k} + \frac{1}{k}. \end{aligned}$$

Hence, for all $k \in \mathbb{N}^*$, $\|q\| \leq \|q - r_{n_k}\| + \|r_{n_k}\| < 2(\frac{1}{1+n_k} + \frac{1}{k})$, which implies that $q = 0$, and so that $(x_n)_n$ separate the points of K. Now, suppose that K is weak* compact subset of X^*. We show that the weak* topology of X^* restricted to K is metrizable. Indeed, each $x \in X$ determines a seminorm ν_x on X^* given by

$$\nu_x(p) = |\langle p, x\rangle|, \quad p \in X^*.$$

The family of seminorms $(\nu_x)_{x\in X}$ induces the weak* topology $\sigma(X^*, X)$ on X^*. The subfamily $(\nu_{x_n})_n$ also induces a topology on X^*, which we will call $\mathcal{T}$. Since this is a smaller family of seminorms, we have $\mathcal{T} \subseteq \sigma(X^*, X)$. Suppose that $p, p' \in K$ and $\nu_{x_n}(p - p') = 0$ for all $n \in \mathbb{N}$. Then we have $\langle p, x_n\rangle = \langle p', x_n\rangle$ for all $n \in \mathbb{N}$ and so we have that $p = p'$ since $(x_n)_n$ separates the points of K. Consequently, K is Hausdorff with respect to the topology $\mathcal{T}_{|K}$ (the restriction of $\mathcal{T}$ to K). Thus $\mathcal{T}_{|K}$ is a Hausdorff topology on K induced from a countable family of seminorms, so this topology is metrizable. More precisely, $\mathcal{T}_{|K}$ is induced from the metric

$$d(p, p') := \sum_{n=0}^{+\infty} 2^{-n} \frac{\nu_{x_n}(p - p')}{1 + \nu_{x_n}(p - p')}.$$

Then we have that K is Hausdorff with respect to $\mathcal{T}_{|K}$, and is compact with respect to $\sigma(X^*, X)_{|K}$. Lemma 1 implies that $\mathcal{T}_{|K} = \sigma(X^*, X)_{|K}$. Hence $\sigma(X^*, X)_{|K}$ is metrizable.

(2) Let $(p_n)_n$ be a weak* null sequence in X^* and set $K = \overline{co}^{w^*}\{p_n : n \in \mathbb{N}\}$. Clearly K is a convex and weak* compact subset of X^*. According to Haydon's theorem [[8], Theorem 3.3] the weak* compact convex set K is the norm closed convex hull of its extreme points whenever $ex(K)$ (the set of extreme points of K) is norm separable. By the Milman theorem [[10], p.9] $ex(K) \subset \overline{\{p_n : n \in \mathbb{N}\}}^{w^*} = \{p_n : n \in \mathbb{N}\} \cup \{0\}$ so that $ex(K)$ is norm separable and, hence, by Haydon's theorem, K itself is weak* compact, convex, and norm separable. $\square$

The following proposition will be used in the proof of Theorem 3.

Proposition 1. *Let X be a Banach space and K be a weak* compact and norm separable subset of X^* containing* 0*. Then, there exists a continuous seminorm h on X^* satisfying*

(1) *h is weak* lower semicontinuous and sequentially weak* continuous,*

(2) *the restriction $h_{|K}$ of h to K has a weak*-strong minimum at* 0.

Proof. Using Lemma 2, there exists a sequence $(x_k)_k \subset S_X$ which separate the points of K. Define the function $h : X^* \longrightarrow \mathbb{R}$ as follows:

$$h(x^*) = (\sum_{k\geq 0} 2^{-k}(\langle x^*, x_k\rangle)^2)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$

It is clear that h is a seminorm, and since $h(x^*) \leq \|x^*\|$ for all $x^* \in X^*$, it is also continuous. Since h is the supremum of a sequence of weak* continuous functions, it is weak* lower semicontinuous. On the other hand, since the series $\sum_{k\geq 0} 2^{-k}(\langle x^*, x_k\rangle)^2$ uniformly converges on bounded sets of X^* and since the maps $\hat{x}_k : x^* \mapsto \langle x^*, x_k\rangle$ are weak* continuous for all $k \in \mathbb{N}$, then h is sequentially weak* continuous. If $p \in K$ and $h(p) = 0$, then $\langle p, x_k\rangle = 0$ for all $k \in \mathbb{N}$ which implies that $p = 0$, since the sequence $(x_k)_k$ separate the points of K. Hence, the restriction of h to K has a unique minimum at 0. This minimum is necessarily a weak*-strong minimum since K is weak* metrizable by Lemma 2, this follows from a general fact which say that for every lower semicontinuous function on a compact metric space (K, d), a unique minimum is necessarily a strong minimum for the metric d in question. □

3 Limited operators and differentiability.

Recall that the domain of a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, is the set

$$dom(f) := \{x \in X / f(x) < +\infty\}.$$

For a function f with $dom(f) \neq \emptyset$, the Fenchel transform of f is defined on the dual space for all $p \in X^*$ by

$$f^*(p) := \sup_{x\in X}\{\langle p, x\rangle - f(x)\}.$$

The second transform $(f^*)^*$ is defined on the bidual X^{**} by the same formula. We denote by f^{**}, the restriction of $(f^*)^*$ to X, where X is identified to a subspace of X^{**} by the canonical embedding. Recall that the Fenchel theorem state that $f = f^{**}$ if and only if f is convex lower semicontinuous on X.

The "if" part of Theorem 1 is given by the following theorem.

Theorem 3. *Let Y and X be Banach spaces and let $T : Y \longrightarrow X$ be a linear continuous operator. Suppose that $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever $f : X \longrightarrow \mathbb{R}$ is convex continuous and Gâteaux differentiable at $T(y) \in Y$. Then T is a limited operator.*

Proof. Let $(p_n)_n$ be a weak* null sequence in X^*. We want to prove that $\|T^*(p_n)\|_{Y^*} \to 0$. Set

$$K = \overline{co}^{w^*}\{p_n : n \in \mathbb{N}\}.$$

According to Lemma 2, K is convex weak* compact and norm separable. Using Proposition 1, there exists a continuous seminorm which is weak* lower semicontinuous and sequentially weak* continuous $h : X^* \longrightarrow \mathbb{R}$ such that the restriction $h_{|K}$ of h to K has a weak*-strong minimum at 0 and in particular $\min_K h = h(0) = 0$. Since the sequence $(p_n)_n$ weak* converges to 0, it follows that $\lim_n h(p_n) = h(0) = \min_K h$. Thus, $(p_n)_n$ is a minimizing sequence for $h_{|K}$. Set $g = h + \delta_K$, where δ_K denotes the indicator function, which is equal to 0 on K and equal to $+\infty$ otherwise. Since K is convex, weak*-closed and norm bounded, then g is a convex and weak* lower semicontinuous function with a norm bounded domain $dom(g) = K$. Moreover we have,

(1) $g(p) > 0 = g(0) = \min_{X^*}(g)$ for all $p \in X^* \setminus \{0\}$.

(2) $\lim_{n\to+\infty} g(p_n) = \min_{X^*}(g)$.

Hence, there exists a convex and Lipschitz continuous function $f : X \longrightarrow \mathbb{R}$ such that $g = f^*$ (we can take $f = g^*_{|X}$). The function f is Gâteaux differentiable at 0 with Gâteaux derivative $\nabla f(0) = 0$, this is due to the fact that $f^* = g$ has a weak*-strong minimum at 0 (we can see [Corollary 1. [2]]). Thus, from our hypothesis, $f \circ T$ is Fréchet differentiable at 0 with Fréchet derivative equal to 0. It follows that $(f \circ T)^*$ has a norm-strong minimum at 0 (see [Corollary 2. [2]]). Now, we prove that $(T^*(p_n))_n$ is a minimizing sequence for $(f \circ T)^*$, which will implies that $\|T^*(p_n)\|_{Y^*} \to 0$. Indeed, on one hand, we have $0 = \min_{X^*}(g) = -g^*(0) = -f(0)$. On the other hand we have

$$\begin{aligned}
0 = -f(0) \leq \sup_{y\in Y}\{-f \circ T(y)\} &:= (f \circ T)^*(0)\\
&\leq \sup_{x\in X}\{-f(x)\}\\
&= f^*(0)\\
&= g(0)\\
&= 0.
\end{aligned}$$

It follows that $(f \circ T)^*(0) = 0$. Hence, since $(f \circ T)^*$ has a minimum at 0, we obtain

$$\begin{aligned}
0 = (f \circ T)^*(0) \leq (f \circ T)^*(T^*(p_n)) &:= \sup_{y\in Y}\{\langle T^*(p_n), y\rangle - f \circ T(y)\}\\
&= \sup_{y\in Y}\{\langle p_n, T(y)\rangle - f(T(y))\}\\
&\leq \sup_{x\in X}\{\langle p_n, x\rangle - f(x)\}\\
&= f^*(p_n)\\
&= g(p_n).
\end{aligned}$$

Since, $g(p_n) \to 0$, it follows that $(f \circ T)^*(T^*(p_n)) \to 0 = (f \circ T)^*(0)$. In other words, $(T^*(p_n))_n$ is a minimizing sequence for $(f \circ T)^*$. Since $(f \circ T)^*$ has a norm-strong minimum at 0, we obtain that $\|T^*(p_n)\|_{Y^*} \to 0$, which implies that T is a limited operator. □

The "only if" part of Theorem 1 is given by the following theorem.

Theorem 4. *Let Y and X be two Banach spaces and $T : Y \longrightarrow X$ be a limited operator. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, be a convex lower semicontinuous function and let $a \in Y$ such that $T(a)$ belongs to the interior of $dom(f)$. Then, $f \circ T$ is Fréchet differentiable at $a \in Y$ with Fréchet-derivative $T^*(Q) \in Y^*$, whenever f is Gâteaux differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$.*

Proof. Since f is convexe lower semicontinuous and $T(a)$ is in the interior of $dom(f)$, there exists $r_a > 0$ and $L_a > 0$ such that f is L_a-Lipschitz continuous on the closed ball $B_X(T(a), r_y)$. It is well known that there exists a convex L_a-Lipschitz continuous function $\tilde{f}_a$ on X such that $\tilde{f}_a = f$ on $B_X(T(a), r_a)$ (See for instance Lemma 2.31 [9]). It follows that $\tilde{f}_a \circ T = f \circ T$ on $B_Y(a, \frac{r_a}{\|T\|})$, since $T(B_X(a, \frac{r_a}{\|T\|}))$ is a subset of $B_X(T(a), r_a)$ (we can assume that $T \neq 0$). Replacing f by $\frac{1}{L_a}\tilde{f}_a$, we can assume without loss of generality that f is convexe 1-Lipschitz continuous on X. It follows that $dom(f^*) \subset B_{X^*}$ (the closed unit ball of X^*).

Claim. Suppose that f is Gâteaux differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$, then the function $q \mapsto f^*(q) - \langle q, T(a)\rangle$ has a weak*-strong minimum on B_{X^*} at Q.

Proof of the claim. See [Corollary 1. [2]].

Now, suppose by contradiction that $T^*(Q)$ is not the the Fréchet derivative of $f \circ T$ at a. There exist $\varepsilon > 0$, $t_n \longrightarrow 0^+$ and $h_n \in Y$, $\|h_n\|_Y = 1$ such that for all $n \in \mathbb{N}^*$,

$$f \circ T(a + t_n h_n) - f \circ T(a) - \langle T^*(Q), t_n h_n\rangle > \varepsilon t_n. \tag{1}$$

Let $r_n = t_n/n$ for all $n \in \mathbb{N}^*$ and choose $p_n \in B_{X^*}$ such that

$$f^*(p_n) - \langle p_n, T(a + t_n h_n)\rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n)\rangle\} + r_n. \tag{2}$$

From (2) we get

$$f^*(p_n) - \langle p_n, T(a)\rangle \quad < \quad \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a)\rangle\} + 2t_n\|T\| + r_n.$$

This implies that the sequence $(p_n)_n$ minimize the function $q \mapsto f^*(q) - \langle q, T(a)\rangle$ on B_{X^*}. Using the claim, the function $q \mapsto f^*(q) - \langle q, T(a)\rangle$ has a weak*-strong minimum on B_{X^*} at Q, it follows that $(p_n)_n$ weak* converges to Q and so (since T is limited) we have

$$\|T^*(p_n - Q)\|_{Y^*} \longrightarrow 0. \tag{3}$$

On the other hand, since $f(T(a + t_n h_n)) = f^{**}(T(a + t_n h_n)) = -\inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n)\rangle\}$, using (2) we obtain for all $y \in Y$

$$\begin{aligned} f \circ T(a + t_n h_n) - \langle p_n, T(a + t_n h_n)\rangle &< -f^*(p_n) + r_n \\ &\leq f \circ T(y) - \langle p_n, T(y)\rangle + r_n. \end{aligned}$$

Replacing y by a in the above inequality we obtain

$$f \circ T(a + t_n h_n) - \langle p_n, T(t_n h_n)\rangle \quad \leq \quad f \circ T(a) + r_n. \tag{4}$$

Combining (1) and (4) we get

$$\begin{aligned} \varepsilon &< \langle p_n, T(h_n)\rangle - \langle T^*(Q), h_n\rangle + r_n/t_n \\ &= \langle T^*(p_n), h_n\rangle - \langle T^*(Q), h_n\rangle + \frac{1}{n} \\ &\leq \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n} \end{aligned}$$

which is a contradiction with (3). Thus $f \circ T$ is Fréchet differentiable at a with Fréchet derivative $T^*(Q)$. □

Now, we give the proof of Theorem 2.

Proof of Theorem 2. (1) $\Longrightarrow$ (2) is the deeper Josefson-Nissenzweig theorem [[7], Chapter XII].

(2) $\Longrightarrow$ (1) is well known.

(2) $\Longrightarrow$ (3) Let $(p_n)_n$ be a weak* null sequence in Y^* such that $\inf_n \|p_n\| > 0$ and set $K = \overline{co}^{w^*}\{p_n : n \in \mathbb{N}\}$. By Lemma 2, the set K is convex norm separable and weak* compact

metrizable. On the other hand, from Proposition 1, there exists a continuous seminorm h which is weak* lower semicontinuous and weak* sequentially continuous on Y^* such that the restriction of h to K has a weak*-strong minimum at 0. It remains to show that 0 is not a norm-strong minimum for $h_{|K}$. Indeed, since $(p_n)_n$ is weak* null and h is weak* sequentially continuous, then $\lim_n h(p_n) = h(0) = \min_K h$. So $(p_n)_n$ is a minimizing sequence for $h_{|K}$ which not converges to 0 since $\inf_n \|p_n\| > 0$. Hence, 0 is not a norm-strong minimum for $h_{|K}$.

(3) $\Longrightarrow$ (2) Since 0 is not a norm-strong minimum for the restriction $h_{|K}$, there exists a sequence $(p_n)_n$ that minimize h on K but $\|p_n\| \nrightarrow 0$. Since $h_{|K}$ has a weak*-strong minimum at 0, it follows that $(p_n)_n$ weak* converges to 0. Hence, $(p_n)_n$ weak* converges to 0 but $\|p_n\| \nrightarrow 0$. Thus, there exists a JN-sequence in Y^*.

(2) $\Longrightarrow$ (4) This part is given by taking $X = Y$ and $T = I$ the identity map. Indeed, there exists a sequence $(p_n)_n$ which weak* converges to 0 but $\inf_n \|I^*(p_n)\| = \inf_n \|p_n\| > 0$. So I cannot be a limited operator.

(4) $\Longrightarrow$ (5). Indeed, if there exists a Banach space X and a non-limited operator $T : Y \longrightarrow X$, by using Theorem 1, there exists a convex continuous function $f : X \longrightarrow \mathbb{R}$ and a point $y \in Y$ such that f is Gâteaux differentiable at $T(y) \in X$ but $f \circ T$ is not Fréchet differentiable at y. So $f \circ T$ is Gâteaux but not Fréchet differentiable at y. Hence, $f \circ T$ is a convex continuous PGNF-function on Y.

(5) $\Longrightarrow$ (2) Let f be a PGNF-function on Y. We can assume without loss of generality that f is Gâteaux differentiable at 0 with Gâteaux-derivative equal to 0, but f is not Fréchet differentiable at 0. It follows from classical duality result (see Corollary 1. and Corollary 2. in [2]) that f^* has a weak*-strong minimum but not norm-strong minimum at 0. Since 0 is not a norm-strong minimum for f^*, there exists a sequence $(p_n)_n \in X^*$ minimizing f^* such that $\|p_n\| \nrightarrow 0$. On the other hand, since f^* has a weak*-strong minimum at 0, and $(p_n)_n$ minimize f^*, we have that $(p_n)_n$ weak* converges to 0. Thus, $(p_n)_n$ weak* converges to 0 but $\|p_n\| \nrightarrow 0$. Hence, there exists a JN-sequence. □

Canonical construction of PGNF-function. There exist different way to build a PGNF-function in infinite dimentional Banach spaces. We can find examples of such constructions in [5]. We present below a different method for constructing a PGNF-function on a Banach space X canonically from a JN-sequence. Given a JN-sequence $(p_n)_n \subset X^*$, we set $K = \overline{co}^{w^*}\{p_n : n \in \mathbb{N}\}$. Using Lemma 2, there exists a sequence $(x_n)_n \in S_X$ which separates the points of K, and as in the proof of Proposition 1, there exist a continuous seminorm h which is weak* lower semicontinuous and weak* sequentially continuous such that $h_{|K}$ has a weak*-strong minimum at 0. The function h is given explicitly as follows

$$h(x^*) = (\sum_{n\geq 0} 2^{-n}(\langle x^*, x_n\rangle)^2)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$

Since $(p_n)_n$ weak* converges to 0, it follows that (p_n) is a minimizing sequence for $h_{|K}$. Since $(p_n)_n$ is a JN-sequence, it follows that 0 is not a norm-strong minimum for $h_{|K}$. Define the function f by

$$f(x) = (h + \delta_K)^*(\hat{x}), \quad \forall x \in X,$$

where δ_K denotes the indicator function, which is equal to 0 on K and equal to $+\infty$ otherwise and where for each $x \in X$, we denote by $\hat{x} \in X^{**}$ the linear map $x^* \mapsto \langle x^*, x\rangle$ for all $x^* \in X^*$. Then f is convex Lipschitz continuous, Gâteaux differentiable at 0 (since $h + \delta_K$ has a weak*-strong minimum) but not Fréchet differentiable at 0 (because 0 is not a norm-strong minimum for $h + \delta_K$).

4 Appendix.

There exists a class of Banach spaces $(E, \|.\|_E)$ such that the canonical embedding $i : E \longrightarrow E^{**}$ is a limited operator. This class contains in particular the space c_0 and any closed subspace

F of c_0 (This class is also stable by product and quotient. For more information see [6]). In this setting, Theorem 4 gives immediately the following corollary.

Corollary 1. *Suppose that the canonical embedding $i : E \longrightarrow E^{**}$ is a limited operator. Let $g : E^{**} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function. Suppose that $x \in E$ belongs to the interior of dom(g) and that g is Gâteaux differentiable at $x \in E$ (we use the identification $i(x) = x$), then the restriction of g to E is Fréchet differentiable at x. In particular, if $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex lower semicontinuous function, $x \in E$ belongs to the interior of dom$((f^*)^*)$ and $(f^*)^*$ is Gâteaux differentiable at x, then f is Fréchet differentiable at x.*

We obtain the following corollary by combining Proposition 2 and a delicate result due to Zajicek (see [Theorem 2; [12]]), which say that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux differentiable, can be covered by countably many $d.c$ (that is, delta-convex) $hypersurface$. Recall that in a separable Banach space Y, each set A which can be covered by countably many $d.c\ hypersurface$ is σ-lower porous, also σ-directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and Γ-null. For details about this notions of small sets we refer to [13] and references therein. Note that a limited set in a separable Banach space is relatively compact [4].

Proposition 2. *Let Y and X be Banach spaces and $T : Y \longrightarrow X$ be a limited operator with a dense range. Let $f : X \longrightarrow \mathbb{R}$ be a convex continuous function. Then $f \circ T$ is Gâteaux differentiable at $a \in Y$ if and only if, $f \circ T$ is Fréchet differentiable at $a \in Y$.*

Proof. Suppose that $f \circ T$ is Gâteaux differentiable at $a \in Y$. It follows that f is Gâteaux differentiable at $T(a)$ with respect to the direction $T(Y)$ which is dense in X. It follows (from a classical fact on locally Lipschitz continuous functions) that f is Gâteaux differentiable at $T(a)$ on X. So by Theorem 4, $f \circ T$ is Fréchet differentiable at $a \in Y$. The converse is always true. □

Corollary 2. *Let Y be a separable Banach space, X be a Banach spaces and $T : Y \longrightarrow X$ be a compact operator with a dense range. Let $f : X \longrightarrow \mathbb{R}$, be a convex and continuous function. Then, the set of all points at which $f \circ T$ is not Fréchet differentiable can be covered by countably many $d.c\ hypersurface$.*

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