



# Limited operators and differentiability

Mohammed Bachir

► **To cite this version:**

| Mohammed Bachir. Limited operators and differentiability. 2016. <hal-01265147v2>

**HAL Id: hal-01265147**

**<https://hal-paris1.archives-ouvertes.fr/hal-01265147v2>**

Submitted on 12 Feb 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Limited operators and differentiability.

Mohammed Bachir

February 12, 2016

*Laboratoire SAMM 4543, Université Paris 1 Panthéon-Sorbonne, Centre P.M.F. 90 rue Tolbiac 75634 Paris cedex 13*

*Email : Mohammed.Bachir@univ-paris1.fr*

**Abstract.** We characterize the limited operators by differentiability of convex continuous functions. Given Banach spaces  $Y$  and  $X$  and a linear continuous operator  $T : Y \rightarrow X$ , we prove that  $T$  is a limited operator if and only if, for every convex continuous function  $f : X \rightarrow \mathbb{R}$  and every point  $y \in Y$ ,  $f \circ T$  is Fréchet differentiable at  $y \in Y$  whenever  $f$  is Gâteaux differentiable at  $T(y) \in X$ .

**Keyword, phrase:** Limited operators, Gâteaux differentiability, Fréchet differentiability, convex functions, extreme points.

**2010 Mathematics Subject:** Primary 46G05, 49J50, 58C20, 46B20, secondary 47B07.

## 1 Introduction.

A subset  $A$  of a Banach space  $X$  is called limited, if every weak\* null sequence  $(p_n)_n$  in  $X^*$  converges uniformly on  $A$ , that is,

$$\lim_{n \rightarrow +\infty} \sup_{x \in A} |\langle p_n, x \rangle| = 0.$$

We know that every relatively compact subset of  $X$  is limited, but the converse is false in general. A bounded linear operator  $T : Y \rightarrow X$  between Banach spaces  $Y$  and  $X$  is called limited, if  $T$  takes the closed unit ball  $B_Y$  of  $Y$  to a limited subset of  $X$ . It is easy to see that  $T : Y \rightarrow X$  is limited if and only if, the adjoint operator  $T^* : X^* \rightarrow Y^*$  takes weak\* null sequence to norm null sequence. For useful properties of limited sets and limited operators we refer to [11], [4], [6] and [1].

We know that in a finite dimensional Banach space, the notions of Gâteaux and Fréchet differentiability coincide for convex continuous functions. In [5], Borwein and Fabian proved that a Banach space  $Y$  is infinite dimensional if and only if, there exists on  $Y$  functions  $f$  having points at which  $f$  is Gâteaux but not Fréchet differentiable. They also pointed in the introduction of [5] the observation that if the sup-norm  $\|\cdot\|_\infty$  on  $c_0$  is Gâteaux differentiable at some point, then it is Fréchet differentiable there. In this article we observe that this phenomenon is not just related to the sup-norm but more generally, for each convex lower semicontinuous function  $g : l^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$ , if  $g$  is Gâteaux differentiable at some point  $a \in c_0$  which is in the interior of its domain, then the restriction of  $g$  to  $c_0$  is Fréchet differentiable at  $a$ . This hold in particular when  $g = (f^*)^*$  is the Fenchel biconjugate of a convex continuous function  $f : c_0 \rightarrow \mathbb{R}$ . In fact, this phenomenon is due, (see Corollary 1 in the Appendix and the comment just before), to the fact that the canonical embedding  $i : c_0 \rightarrow l^\infty$  is a limited operator (see the reference [6]).

The goal of this paper, is to prove the following characterization of limited operators in terms of the coincidence of Gâteaux and Fréchet differentiability of convex continuous functions.

**Theorem 1.** *Let  $Y$  and  $X$  be two Banach spaces and  $T : Y \longrightarrow X$  be a continuous linear operator. Then,  $T$  is a limited operator if and only if, for every convex continuous function  $f : X \longrightarrow \mathbb{R}$  and every  $y \in Y$ , the function  $f \circ T$  is Fréchet differentiable at  $y \in Y$  whenever  $f$  is Gâteaux differentiable at  $T(y) \in X$ .*

As consequence we give, in Theorem 2 below, new characterizations of infinite dimensional Banach spaces, complementing a result of Borwein and Fabian in [[5], Theorem 1.].

A real valued function  $f$  on a Banach space will be called a PGNF-function (see [5]) if there exists a point at which  $f$  is Gâteaux but not Fréchet differentiable. A JN-sequence (due to Josefson-Nissenzweig theorem, see [[7], Chapter XII]) is a sequence  $(p_n)_n$  in a dual space  $Y^*$  that is weak\* null and  $\inf_n \|p_n\| > 0$ . We say that a function  $g$  on  $X^*$  has a norm-strong minimum (resp. weak\*-strong minimum) at  $p \in X^*$  if  $g(p) = \inf_{q \in X^*} g(q)$  and  $(p_n)_n$  norm converges (resp. weak\* converges) to  $p$  whenever  $g(p_n) \longrightarrow g(p)$ . A norm-strong minimum and weak\*-strong minimum are in particular unique.

**Theorem 2.** *Let  $Y$  be a Banach space. Then the following assertions are equivalent.*

- (1)  $Y$  is infinite dimensional.
- (2) There exists a JN-sequence in  $Y^*$ .
- (3) There exists a convex norm separable and weak\* compact metrizable subset  $K$  of  $Y^*$  containing 0 and a continuous seminorm  $h$  on  $X^*$  which is weak\* lower semicontinuous and weak\* sequentially continuous, such that the restriction  $h|_K$  has a weak\*-strong minimum but not norm-strong minimum at 0.
- (4) There exists a Banach space  $X$  and a linear continuous non-limited operator  $T : Y \longrightarrow X$ .
- (5) There exists on  $Y$  a convex continuous PGNF-function.

In Section 2 we give some preliminary results, specially the key Lemma 2. In Section 3, we give the proof of Theorem 1 (divided into two part, Theorem 3 and Theorem 4) and the proof of Theorem 2. In Section 4 we give some complementary remarks.

## 2 Preliminaries.

We recall the following classical result.

**Lemma 1.** *Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be topologies on a set  $K$  such that*

- (1)  $K$  is Hausdorff with respect  $\mathcal{T}_1$ ,
- (2)  $K$  is compact with respect to  $\mathcal{T}_2$ ,
- (3)  $\mathcal{T}_1 \subset \mathcal{T}_2$ .

*Then  $\mathcal{T}_1 = \mathcal{T}_2$ .*

*Proof.* Let  $F \subset K$  be a  $\mathcal{T}_2$ -closed set. It follows that  $F$  is  $\mathcal{T}_2$ -compact, since  $K$  is  $\mathcal{T}_2$ -compact. Let  $\{\mathcal{O}_i : i \in I\}$  be any cover of  $F$  by  $\mathcal{T}_1$ -open sets. Since  $\mathcal{T}_1 \subset \mathcal{T}_2$ , then each of these sets is also  $\mathcal{T}_2$ -open. Hence, there exist a finite subcollection that covers  $F$ . It follows that  $F$  is  $\mathcal{T}_1$ -compact and therefore is  $\mathcal{T}_1$ -closed since  $\mathcal{T}_1$  is Hausdorff. This implies that  $\mathcal{T}_2 \subset \mathcal{T}_1$ . Consequently,  $\mathcal{T}_1 = \mathcal{T}_2$ . □

Now, we establish the following useful lemma. If  $B$  is a subset of a dual Banach space  $X^*$ , we denote by  $\overline{co}^{w^*}(B)$  the weak\* closed convex hull of  $B$ .

**Lemma 2.** *Let  $X$  be a Banach space and  $K$  be a subset of  $X^*$ .*

(1) *Suppose that  $K$  is norm separable, then there exists a sequence  $(x_n)_n$  in the unit sphere  $S_X$  of  $X$  which separate the points of  $K$  i.e. for all  $p, p' \in K$ , if  $\langle p, x_n \rangle = \langle p', x_n \rangle$  for all  $n \in \mathbb{N}$ , then  $p = p'$ . Consequently, if  $K$  is a weak\* compact and norm separable set of  $X^*$ , then the weak\* topology of  $X^*$  restricted to  $K$  is metrizable.*

(2) *Let  $(p_n)_n$  be a weak\* null sequence in  $X^*$ . Then, the set  $\overline{co}^{w^*} \{p_n : n \in \mathbb{N}\}$  is convex weak\* compact and norm separable.*

*Proof.* (1) Since  $K$  is norm separable, then  $K - K := \{a - b / (a, b) \in K \times K\}$  is also norm separable and so there exists a sequence  $(q_n)_n$  of  $K - K$  which is dense in  $K - K$ . According to the Bishop-Phelps theorem [3], the set

$$D = \{r \in X^* \mid r \text{ attains its supremum on the sphere } S_X\}$$

is norm-dense in the dual  $X^*$ . Thus, for each  $n \in \mathbb{N}$ , there exists  $r_n \in D$  such that  $\|q_n - r_n\| < \frac{1}{1+n}$ . For each  $n \in \mathbb{N}$ , let  $x_n \in S_X$  be such that  $\|r_n\| = \langle r_n, x_n \rangle$ . We claim that the sequence  $(x_n)_n$  separate the points of  $K$ . Indeed, let  $q \in K - K$  and suppose that  $\langle q, x_n \rangle = 0$ , for all  $n \in \mathbb{N}$ . There exists a subsequence  $(q_{n_k})_k \subset K - K$  such that  $\|q_{n_k} - q\| < \frac{1}{k}$  for all  $k \in \mathbb{N}^*$  and so we have  $\|r_{n_k} - q\| < \frac{1}{1+n_k} + \frac{1}{k}$ . It follows that

$$\begin{aligned} \|r_{n_k}\| &= \langle r_{n_k}, x_{n_k} \rangle \\ &= \langle r_{n_k}, x_{n_k} \rangle - \langle q, x_{n_k} \rangle \\ &\leq \|r_{n_k} - q\| \\ &< \frac{1}{1+n_k} + \frac{1}{k}. \end{aligned}$$

Hence, for all  $k \in \mathbb{N}^*$ ,  $\|q\| \leq \|q - r_{n_k}\| + \|r_{n_k}\| < 2(\frac{1}{1+n_k} + \frac{1}{k})$ , which implies that  $q = 0$ , and so that  $(x_n)_n$  separate the points of  $K$ . Now, suppose that  $K$  is weak\* compact subset of  $X^*$ . We show that the weak\* topology of  $X^*$  restricted to  $K$  is metrizable. Indeed, each  $x \in X$  determines a seminorm  $\nu_x$  on  $X^*$  given by

$$\nu_x(p) = |\langle p, x \rangle|, \quad p \in X^*.$$

The family of seminorms  $(\nu_x)_{x \in X}$  induces the weak\* topology  $\sigma(X^*, X)$  on  $X^*$ . The subfamily  $(\nu_{x_n})_n$  also induces a topology on  $X^*$ , which we will call  $\mathcal{T}$ . Since this is a smaller family of seminorms, we have  $\mathcal{T} \subseteq \sigma(X^*, X)$ . Suppose that  $p, p' \in K$  and  $\nu_{x_n}(p - p') = 0$  for all  $n \in \mathbb{N}$ . Then we have  $\langle p, x_n \rangle = \langle p', x_n \rangle$  for all  $n \in \mathbb{N}$  and so we have that  $p = p'$  since  $(x_n)_n$  separates the points of  $K$ . Consequently,  $K$  is Hausdorff with respect to the topology  $\mathcal{T}|_K$  (the restriction of  $\mathcal{T}$  to  $K$ ). Thus  $\mathcal{T}|_K$  is a Hausdorff topology on  $K$  induced from a countable family of seminorms, so this topology is metrizable. More precisely,  $\mathcal{T}|_K$  is induced from the metric

$$d(p, p') := \sum_{n=0}^{+\infty} 2^{-n} \frac{\nu_{x_n}(p - p')}{1 + \nu_{x_n}(p - p')}.$$

Then we have that  $K$  is Hausdorff with respect to  $\mathcal{T}|_K$ , and is compact with respect to  $\sigma(X^*, X)|_K$ . Lemma 1 implies that  $\mathcal{T}|_K = \sigma(X^*, X)|_K$ . Hence  $\sigma(X^*, X)|_K$  is metrizable.

(2) Let  $(p_n)_n$  be a weak\* null sequence in  $X^*$  and set  $K = \overline{co}^{w^*} \{p_n : n \in \mathbb{N}\}$ . Clearly  $K$  is a convex and weak\* compact subset of  $X^*$ . According to Haydon's theorem [[8], Theorem 3.3] the weak\* compact convex set  $K$  is the norm closed convex hull of its extreme points whenever  $ex(K)$  (the set of extreme points of  $K$ ) is norm separable. By the Milman theorem [[10], p.9]  $ex(K) \subset \overline{\{p_n : n \in \mathbb{N}\}}^{w^*} = \{p_n : n \in \mathbb{N}\} \cup \{0\}$  so that  $ex(K)$  is norm separable and, hence, by Haydon's theorem,  $K$  itself is weak\* compact, convex, and norm separable.  $\square$

The following proposition will be used in the proof of Theorem 3.

**Proposition 1.** *Let  $X$  be a Banach space and  $K$  be a weak\* compact and norm separable subset of  $X^*$  containing 0. Then, there exists a continuous seminorm  $h$  on  $X^*$  satisfying*

- (1)  *$h$  is weak\* lower semicontinuous and sequentially weak\* continuous,*
- (2) *the restriction  $h|_K$  of  $h$  to  $K$  has a weak\*-strong minimum at 0.*

*Proof.* Using Lemma 2, there exists a sequence  $(x_k)_k \subset S_X$  which separate the points of  $K$ . Define the function  $h : X^* \rightarrow \mathbb{R}$  as follows:

$$h(x^*) = \left( \sum_{k \geq 0} 2^{-k} (\langle x^*, x_k \rangle)^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$

It is clear that  $h$  is a seminorm, and since  $h(x^*) \leq \|x^*\|$  for all  $x^* \in X^*$ , it is also continuous. Since  $h$  is the supremum of a sequence of weak\* continuous functions, it is weak\* lower semicontinuous. On the other hand, since the series  $\sum_{k \geq 0} 2^{-k} (\langle x^*, x_k \rangle)^2$  uniformly converges on bounded sets of  $X^*$  and since the maps  $\hat{x}_k : x^* \mapsto \langle x^*, x_k \rangle$  are weak\* continuous for all  $k \in \mathbb{N}$ , then  $h$  is sequentially weak\* continuous. If  $p \in K$  and  $h(p) = 0$ , then  $\langle p, x_k \rangle = 0$  for all  $k \in \mathbb{N}$  which implies that  $p = 0$ , since the sequence  $(x_k)_k$  separate the points of  $K$ . Hence, the restriction of  $h$  to  $K$  has a unique minimum at 0. This minimum is necessarily a weak\*-strong minimum since  $K$  is weak\* metrizable by Lemma 2, this follows from a general fact which says that for every lower semicontinuous function on a compact metric space  $(K, d)$ , a unique minimum is necessarily a strong minimum for the metric  $d$  in question.  $\square$

### 3 Limited operators and differentiability.

Recall that the domain of a function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , is the set

$$\text{dom}(f) := \{x \in X / f(x) < +\infty\}.$$

For a function  $f$  with  $\text{dom}(f) \neq \emptyset$ , the Fenchel transform of  $f$  is defined on the dual space for all  $p \in X^*$  by

$$f^*(p) := \sup_{x \in X} \{ \langle p, x \rangle - f(x) \}.$$

The second transform  $(f^*)^*$  is defined on the bidual  $X^{**}$  by the same formula. We denote by  $f^{**}$ , the restriction of  $(f^*)^*$  to  $X$ , where  $X$  is identified to a subspace of  $X^{**}$  by the canonical embedding. Recall that the Fenchel theorem states that  $f = f^{**}$  if and only if  $f$  is convex lower semicontinuous on  $X$ .

The "if" part of Theorem 1 is given by the following theorem.

**Theorem 3.** *Let  $Y$  and  $X$  be Banach spaces and let  $T : Y \rightarrow X$  be a linear continuous operator. Suppose that  $f \circ T$  is Fréchet differentiable at  $y \in Y$  whenever  $f : X \rightarrow \mathbb{R}$  is convex continuous and Gâteaux differentiable at  $T(y) \in X$ . Then  $T$  is a limited operator.*

*Proof.* Let  $(p_n)_n$  be a weak\* null sequence in  $X^*$ . We want to prove that  $\|T^*(p_n)\|_{Y^*} \rightarrow 0$ . Set

$$K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}.$$

According to Lemma 2,  $K$  is convex weak\* compact and norm separable. Using Proposition 1, there exists a continuous seminorm which is weak\* lower semicontinuous and sequentially weak\* continuous  $h : X^* \rightarrow \mathbb{R}$  such that the restriction  $h|_K$  of  $h$  to  $K$  has a weak\*-strong minimum at 0 and in particular  $\min_K h = h(0) = 0$ . Since the sequence  $(p_n)_n$  weak\* converges to 0, it follows that  $\lim_n h(p_n) = h(0) = \min_K h$ . Thus,  $(p_n)_n$  is a minimizing sequence for  $h|_K$ . Set  $g = h + \delta_K$ , where  $\delta_K$  denotes the indicator function, which is equal to 0 on  $K$  and equal to  $+\infty$  otherwise. Since  $K$  is convex, weak\*-closed and norm bounded, then  $g$  is a convex and

weak\* lower semicontinuous function with a norm bounded domain  $\text{dom}(g) = K$ . Moreover we have,

- (1)  $g(p) > 0 = g(0) = \min_{X^*}(g)$  for all  $p \in X^* \setminus \{0\}$ .
- (2)  $\lim_{n \rightarrow +\infty} g(p_n) = \min_{X^*}(g)$ .

Hence, there exists a convex and Lipschitz continuous function  $f : X \rightarrow \mathbb{R}$  such that  $g = f^*$  (we can take  $f = g|_X$ ). The function  $f$  is Gâteaux differentiable at 0 with Gâteaux derivative  $\nabla f(0) = 0$ , this is due to the fact that  $f^* = g$  has a weak\*-strong minimum at 0 (we can see [Corollary 1. [2]]). Thus, from our hypothesis,  $f \circ T$  is Fréchet differentiable at 0 with Fréchet derivative equal to 0. It follows that  $(f \circ T)^*$  has a norm-strong minimum at 0 (see [Corollary 2. [2]]). Now, we prove that  $(T^*(p_n))_n$  is a minimizing sequence for  $(f \circ T)^*$ , which will implies that  $\|T^*(p_n)\|_{Y^*} \rightarrow 0$ . Indeed, on one hand, we have  $0 = \min_{X^*}(g) = -g^*(0) = -f(0)$ . On the other hand we have

$$\begin{aligned} 0 = -f(0) &\leq \sup_{y \in Y} \{-f \circ T(y)\} &:= (f \circ T)^*(0) \\ &\leq \sup_{x \in X} \{-f(x)\} \\ &= f^*(0) \\ &= g(0) \\ &= 0. \end{aligned}$$

It follows that  $(f \circ T)^*(0) = 0$ . Hence, since  $(f \circ T)^*$  has a minimum at 0, we obtain

$$\begin{aligned} 0 = (f \circ T)^*(0) &\leq (f \circ T)^*(T^*(p_n)) &:= \sup_{y \in Y} \{\langle T^*(p_n), y \rangle - f \circ T(y)\} \\ &= \sup_{y \in Y} \{\langle p_n, T(y) \rangle - f(T(y))\} \\ &\leq \sup_{x \in X} \{\langle p_n, x \rangle - f(x)\} \\ &= f^*(p_n) \\ &= g(p_n). \end{aligned}$$

Since,  $g(p_n) \rightarrow 0$ , it follows that  $(f \circ T)^*(T^*(p_n)) \rightarrow 0 = (f \circ T)^*(0)$ . In other words,  $(T^*(p_n))_n$  is a minimizing sequence for  $(f \circ T)^*$ . Since  $(f \circ T)^*$  has a norm-strong minimum at 0, we obtain that  $\|T^*(p_n)\|_{Y^*} \rightarrow 0$ , which implies that  $T$  is a limited operator.  $\square$

The "only if" part of Theorem 1 is given by the following theorem.

**Theorem 4.** *Let  $Y$  and  $X$  be two Banach spaces and  $T : Y \rightarrow X$  be a limited operator. Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , be a convex lower semicontinuous function and let  $a \in Y$  such that  $T(a)$  belongs to the interior of  $\text{dom}(f)$ . Then,  $f \circ T$  is Fréchet differentiable at  $a \in Y$  with Fréchet-derivative  $T^*(Q) \in Y^*$ , whenever  $f$  is Gâteaux differentiable at  $T(a) \in X$  with Gâteaux-derivative  $Q \in X^*$ .*

*Proof.* Since  $f$  is convex lower semicontinuous and  $T(a)$  is in the interior of  $\text{dom}(f)$ , there exists  $r_a > 0$  and  $L_a > 0$  such that  $f$  is  $L_a$ -Lipschitz continuous on the closed ball  $B_X(T(a), r_a)$ . It is well known that there exists a convex  $L_a$ -Lipschitz continuous function  $\tilde{f}_a$  on  $X$  such that  $\tilde{f}_a = f$  on  $B_X(T(a), r_a)$  (See for instance Lemma 2.31 [9]). It follows that  $\tilde{f}_a \circ T = f \circ T$  on  $B_Y(a, \frac{r_a}{\|T\|})$ , since  $T(B_X(a, \frac{r_a}{\|T\|}))$  is a subset of  $B_X(T(a), r_a)$  (we can assume that  $T \neq 0$ ). Replacing  $f$  by  $\frac{1}{L_a} \tilde{f}_a$ , we can assume without loss of generality that  $f$  is convex 1-Lipschitz continuous on  $X$ . It follows that  $\text{dom}(f^*) \subset B_{X^*}$  (the closed unit ball of  $X^*$ ).

*Claim.* Suppose that  $f$  is Gâteaux differentiable at  $T(a) \in X$  with Gâteaux-derivative  $Q \in X^*$ , then the function  $q \mapsto f^*(q) - \langle q, T(a) \rangle$  has a weak\*-strong minimum on  $B_{X^*}$  at  $Q$ .

*Proof of the claim.* See [Corollary 1. [2]].

Now, suppose by contradiction that  $T^*(Q)$  is not the the Fréchet derivative of  $f \circ T$  at  $a$ . There exist  $\varepsilon > 0$ ,  $t_n \rightarrow 0^+$  and  $h_n \in Y$ ,  $\|h_n\|_Y = 1$  such that for all  $n \in \mathbb{N}^*$ ,

$$f \circ T(a + t_n h_n) - f \circ T(a) - \langle T^*(Q), t_n h_n \rangle > \varepsilon t_n. \quad (1)$$

Let  $r_n = t_n/n$  for all  $n \in \mathbb{N}^*$  and choose  $p_n \in B_{X^*}$  such that

$$f^*(p_n) - \langle p_n, T(a + t_n h_n) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\} + r_n. \quad (2)$$

From (2) we get

$$f^*(p_n) - \langle p_n, T(a) \rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a) \rangle\} + 2t_n \|T\| + r_n.$$

This implies that the sequence  $(p_n)_n$  minimize the function  $q \mapsto f^*(q) - \langle q, T(a) \rangle$  on  $B_{X^*}$ . Using the claim, the function  $q \mapsto f^*(q) - \langle q, T(a) \rangle$  has a weak\*-strong minimum on  $B_{X^*}$  at  $Q$ , it follows that  $(p_n)_n$  weak\* converges to  $Q$  and so (since  $T$  is limited) we have

$$\|T^*(p_n - Q)\|_{Y^*} \rightarrow 0. \quad (3)$$

On the other hand, since  $f(T(a + t_n h_n)) = f^{**}(T(a + t_n h_n)) = -\inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n) \rangle\}$ , using (2) we obtain for all  $y \in Y$

$$\begin{aligned} f \circ T(a + t_n h_n) - \langle p_n, T(a + t_n h_n) \rangle &< -f^*(p_n) + r_n \\ &\leq f \circ T(y) - \langle p_n, T(y) \rangle + r_n. \end{aligned}$$

Replacing  $y$  by  $a$  in the above inequality we obtain

$$f \circ T(a + t_n h_n) - \langle p_n, T(t_n h_n) \rangle \leq f \circ T(a) + r_n. \quad (4)$$

Combining (1) and (4) we get

$$\begin{aligned} \varepsilon &< \langle p_n, T(h_n) \rangle - \langle T^*(Q), h_n \rangle + r_n/t_n \\ &= \langle T^*(p_n), h_n \rangle - \langle T^*(Q), h_n \rangle + \frac{1}{n} \\ &\leq \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n} \end{aligned}$$

which is a contradiction with (3). Thus  $f \circ T$  is Fréchet differentiable at  $a$  with Fréchet derivative  $T^*(Q)$ .  $\square$

Now, we give the proof of Theorem 2.

**Proof of Theorem 2.** (1)  $\implies$  (2) is the deeper Josefson-Nissenzweig theorem [[7], Chapter XII].

(2)  $\implies$  (1) is well known.

(2)  $\implies$  (3) Let  $(p_n)_n$  be a weak\* null sequence in  $Y^*$  such that  $\inf_n \|p_n\| > 0$  and set  $K = \overline{co}^{w^*} \{p_n : n \in \mathbb{N}\}$ . By Lemma 2, the set  $K$  is convex norm separable and weak\* compact

metrizable. On the other hand, from Proposition 1, there exists a continuous seminorm  $h$  which is weak\* lower semicontinuous and weak\* sequentially continuous on  $Y^*$  such that the restriction of  $h$  to  $K$  has a weak\*-strong minimum at 0. It remains to show that 0 is not a norm-strong minimum for  $h|_K$ . Indeed, since  $(p_n)_n$  is weak\* null and  $h$  is weak\* sequentially continuous, then  $\lim_n h(p_n) = h(0) = \min_K h$ . So  $(p_n)_n$  is a minimizing sequence for  $h|_K$  which not converges to 0 since  $\inf_n \|p_n\| > 0$ . Hence, 0 is not a norm-strong minimum for  $h|_K$ .

(3)  $\implies$  (2) Since 0 is not a norm-strong minimum for the restriction  $h|_K$ , there exists a sequence  $(p_n)_n$  that minimize  $h$  on  $K$  but  $\|p_n\| \not\rightarrow 0$ . Since  $h|_K$  has a weak\*-strong minimum at 0, it follows that  $(p_n)_n$  weak\* converges to 0. Hence,  $(p_n)_n$  weak\* converges to 0 but  $\|p_n\| \not\rightarrow 0$ . Thus, there exists a JN-sequence in  $Y^*$ .

(2)  $\implies$  (4) This part is given by taking  $X = Y$  and  $T = I$  the identity map. Indeed, there exists a sequence  $(p_n)_n$  which weak\* converges to 0 but  $\inf_n \|I^*(p_n)\| = \inf_n \|p_n\| > 0$ . So  $I$  cannot be a limited operator.

(4)  $\implies$  (5). Indeed, if there exists a Banach space  $X$  and a non-limited operator  $T : Y \rightarrow X$ , by using Theorem 1, there exists a convex continuous function  $f : X \rightarrow \mathbb{R}$  and a point  $y \in Y$  such that  $f$  is Gâteaux differentiable at  $T(y) \in X$  but  $f \circ T$  is not Fréchet differentiable at  $y$ . So  $f \circ T$  is Gâteaux but not Fréchet differentiable at  $y$ . Hence,  $f \circ T$  is a convex continuous PGNF-function on  $Y$ .

(5)  $\implies$  (2) Let  $f$  be a PGNF-function on  $Y$ . We can assume without loss of generality that  $f$  is Gâteaux differentiable at 0 with Gâteaux-derivative equal to 0, but  $f$  is not Fréchet differentiable at 0. It follows from classical duality result (see Corollary 1. and Corollary 2. in [2]) that  $f^*$  has a weak\*-strong minimum but not norm-strong minimum at 0. Since 0 is not a norm-strong minimum for  $f^*$ , there exists a sequence  $(p_n)_n \in X^*$  minimizing  $f^*$  such that  $\|p_n\| \not\rightarrow 0$ . On the other hand, since  $f^*$  has a weak\*-strong minimum at 0, and  $(p_n)_n$  minimize  $f^*$ , we have that  $(p_n)_n$  weak\* converges to 0. Thus,  $(p_n)_n$  weak\* converges to 0 but  $\|p_n\| \not\rightarrow 0$ . Hence, there exists a JN-sequence.  $\square$

**Canonical construction of PGNF-function.** There exist different way to build a PGNF-function in infinite dimensional Banach spaces. We can find examples of such constructions in [5]. We present below a different method for constructing a PGNF-function on a Banach space  $X$  canonically from a JN-sequence. Given a JN-sequence  $(p_n)_n \subset X^*$ , we set  $K = \overline{\text{co}}^{w^*} \{p_n : n \in \mathbb{N}\}$ . Using Lemma 2, there exists a sequence  $(x_n)_n \in S_X$  which separates the points of  $K$ , and as in the proof of Proposition 1, there exist a continuous seminorm  $h$  which is weak\* lower semicontinuous and weak\* sequentially continuous such that  $h|_K$  has a weak\*-strong minimum at 0. The function  $h$  is given explicitly as follows

$$h(x^*) = \left( \sum_{n \geq 0} 2^{-n} (\langle x^*, x_n \rangle)^2 \right)^{\frac{1}{2}}, \quad \forall x^* \in X^*.$$

Since  $(p_n)_n$  weak\* converges to 0, it follows that  $(p_n)$  is a minimizing sequence for  $h|_K$ . Since  $(p_n)_n$  is a JN-sequence, it follows that 0 is not a norm-strong minimum for  $h|_K$ . Define the function  $f$  by

$$f(x) = (h + \delta_K)^*(\hat{x}), \quad \forall x \in X,$$

where  $\delta_K$  denotes the indicator function, which is equal to 0 on  $K$  and equal to  $+\infty$  otherwise and where for each  $x \in X$ , we denote by  $\hat{x} \in X^{**}$  the linear map  $x^* \mapsto \langle x^*, x \rangle$  for all  $x^* \in X^*$ . Then  $f$  is convex Lipschitz continuous, Gâteaux differentiable at 0 (since  $h + \delta_K$  has a weak\*-strong minimum) but not Fréchet differentiable at 0 (because 0 is not a norm-strong minimum for  $h + \delta_K$ ).

## 4 Appendix.

There exists a class of Banach spaces  $(E, \|\cdot\|_E)$  such that the canonical embedding  $i : E \rightarrow E^{**}$  is a limited operator. This class contains in particular the space  $c_0$  and any closed subspace



$F$  of  $c_0$  (This class is also stable by product and quotient. For more information see [6]). In this setting, Theorem 4 gives immediately the following corollary.

**Corollary 1.** *Suppose that the canonical embedding  $i : E \rightarrow E^{**}$  is a limited operator. Let  $g : E^{**} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex lower semicontinuous function. Suppose that  $x \in E$  belongs to the interior of  $\text{dom}(g)$  and that  $g$  is Gâteaux differentiable at  $x \in E$  (we use the identification  $i(x) = x$ ), then the restriction of  $g$  to  $E$  is Fréchet differentiable at  $x$ . In particular, if  $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex lower semicontinuous function,  $x \in E$  belongs to the interior of  $\text{dom}((f^*)^*)$  and  $(f^*)^*$  is Gâteaux differentiable at  $x$ , then  $f$  is Fréchet differentiable at  $x$ .*

We obtain the following corollary by combining Proposition 2 and a delicate result due to Zajicek (see [Theorem 2; [12]]), which say that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux differentiable, can be covered by countably many *d.c* (that is, delta-convex) *hypersurface*. Recall that in a separable Banach space  $Y$ , each set  $A$  which can be covered by countably many *d.c hypersurface* is  $\sigma$ -lower porous, also  $\sigma$ -directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and  $\Gamma$ -null. For details about this notions of small sets we refer to [13] and references therein. Note that a limited set in a separable Banach space is relatively compact [4].

**Proposition 2.** *Let  $Y$  and  $X$  be Banach spaces and  $T : Y \rightarrow X$  be a limited operator with a dense range. Let  $f : X \rightarrow \mathbb{R}$  be a convex continuous function. Then  $f \circ T$  is Gâteaux differentiable at  $a \in Y$  if and only if,  $f \circ T$  is Fréchet differentiable at  $a \in Y$ .*

*Proof.* Suppose that  $f \circ T$  is Gâteaux differentiable at  $a \in Y$ . It follows that  $f$  is Gâteaux differentiable at  $T(a)$  with respect to the direction  $T(Y)$  which is dense in  $X$ . It follows (from a classical fact on locally Lipschitz continuous functions) that  $f$  is Gâteaux differentiable at  $T(a)$  on  $X$ . So by Theorem 4,  $f \circ T$  is Fréchet differentiable at  $a \in Y$ . The converse is always true.  $\square$

**Corollary 2.** *Let  $Y$  be a separable Banach space,  $X$  be a Banach spaces and  $T : Y \rightarrow X$  be a compact operator with a dense range. Let  $f : X \rightarrow \mathbb{R}$ , be a convex and continuous function. Then, the set of all points at which  $f \circ T$  is not Fréchet differentiable can be covered by countably many *d.c hypersurface*.*

## References

- [1] K. T. Andrews, *Dunford-Pettis sets in the space of Bochner integrable functions*, Math. Ann. 241, (1979), 35-41.
- [2] E. Asplund and R. T. Rockafellar, *Gradients of convex functions*. Trans. Amer. Math. Soc. 139, (1969), 443-467.
- [3] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67, (1961), 97-98 .
- [4] J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. 119, (1984), 55-58.
- [5] J. M. Borwein and M. Fabian, *On convex functions having points of Gâteaux differentiability which are not points of Féchet-differentiability*. Can. J. Math. Vol.45 (6), 1993, 1121-1134.
- [6] H. Carrión, P. Galindo, and M.L Lourenco, *Banach spaces whose bounded sets are bounding in the bidual* Annales Academiae Scientiarum Fennicae Mathematica Volumen 31, (2006), 61-70.

- [7] J. Diestel, *Sequences and series in Banach spaces*, Graduate texts in Mathematics, Springer Verlag, N.Y., Berlin, Tokyo, 1984.
- [8] R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford Ser. 27 (1976), 379-385.
- [9] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*. Lecture Notes in Mathematics 1364, Springer-Verlag, Berlin (1993).
- [10] R. R. Phelps, *Lectures on Chaquet's theorem*, Van Nostrand, Princeton, N. J., 1966.
- [11] Th. Schlumprecht, *Limited sets in Banach spaces*, Dissertation, München, 1987.
- [12] L. Zajicek, *On the differentiation of convex functions in finite and infinite dimensional spaces*, Czechoslovak Math. J. 29, (1979), no. 3, 340-348.
- [13] L. Zajicek, *On sigma-porous sets in abstract spaces*, Abstract and Applied Analysis, vol. (2005), issue 5, 509-534,