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INTRODUCTION TO THE PROBABILISTIC 1-LIPSCHITZ MAPS

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Abstract. We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

54E70, 47S50, 46S50

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1. Introduction

We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane (see [2]) on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

A distribution function is a function $F : [-\infty, +\infty] \rightarrow [0, 1]$ which is non-decreasing and left-continuous with $F(-\infty) = 0$; $F(+\infty) = 1$. The set of all distribution functions will be denoted by $\Delta$. The subset of $\Delta$ consisting on distributions $F$ such that $F(0) = 0$ will be denoted by $\Delta^+$. For $F, G \in \Delta^+$, the relation $F \leq G$ is meant by $F(t) \leq G(t)$, for all $t \in \mathbb{R}$. For all $a \in \mathbb{R}$, the distribution $H_a$ is defined as follow

$$H_a(t) = \begin{cases} 
  0, & \text{if } t \leq a \\
  1, & \text{if } t > a 
\end{cases}$$

For $a = +\infty$,

$$H_{\infty}(t) = \begin{cases} 
  0, & \text{if } t \in [-\infty, +\infty[ \\
  1, & \text{if } t = +\infty 
\end{cases}$$
It is well known that \((\Delta, \leq)\) and \((\Delta^+, \leq)\) are complete lattice with the minimal element \(H_\infty\) and the maximal element \(H_0\) (see [1]). Thus, for any nonempty set \(I\) and any family \((F_i)_{i \in I}\) of distributions in \(\Delta\) (resp. in \(\Delta^+\)), the function \(F = \sup_{i \in I} F_i\) is also an element of \(\Delta\) (resp. of \(\Delta^+\)).

**Axioms 1.1.** In this work, we assume that \(\Delta^+\) is equipped with a law \(*\) (a trialngual function) satisfying the following axioms:

(i) \(F * L \in \Delta^+\) for all \(F, L \in \Delta^+\).

(ii) \(F * L = L * F\) for all \(F, L \in \Delta^+\).

(iii) \(F * (L * K) = (F * L) * K\), for all \(F, L, K \in \Delta^+\).

(iv) \(F * H_0 = F\) for all \(F \in \Delta^+\).

(v) \(F \leq L \implies F * K \leq L * K\) for all \(F, L, K \in \Delta^+\).

(vi) Let \(I\) be a set, \((F_i)_{i \in I}\) a family of distributions in \(\Delta^+\) and \(L \in \Delta^+\). Then, \(\sup_{i \in I} (F_i * L) = (\sup_{i \in I} (F_i)) * L\).

**Definition 1.2.** We say that \(*\) is continuous at \((F, L) \in \Delta^+ \times \Delta^+\) if \(\lim_{n \to +\infty} (F_n * L_n)(t) = (F * L)(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(F * L\), whenever \(\lim_{n \to +\infty} F_n(t) = F(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(F\) and \(\lim_{n \to +\infty} L_n(t) = L(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(L\).

**Example 1.3.** (see [3, 1]) Let \(T : [0, 1] \times [0, 1] \to [0, 1]\) be a left-continuous \(t\)-norm, then the operation \(*\) defined for all \(F, L \in \Delta^+\) and for all \(t \in \mathbb{R}\) by

\[
(F * L)(t) = \sup_{s + u = t} T(F(s), L(u))
\]

is continuous at each point \((F, L) \in \Delta^+ \times \Delta^+\) and satisfies the axioms (i)-(vi).

In all this work, we assume that \(\Delta^+\) is equipped with a continuous law \(*\) satisfying the axioms (i)-(vi).

**Definition 1.4.** Let \(G\) be a set and let \(D : G \times G \to (\Delta^+, *, \leq)\) be a map. We say that \((G, D, *)\) is a probabilistic metric space if the following axioms (i)-(iii) hold:

(i) \(D(p, q) = H_0\) iff \(p = q\).

(ii) \(D(p, q) = D(q, p)\) for all \(p, q \in G\)

(iii) \(D(p, q) * D(q, r) \leq D(p, r)\) for all \(p, q, r \in G\)

2. **Probabilistic 1-Lipschitz Maps**

We define probabilistic continuous functions.

**Definition 2.1.** Let \((G, D, *)\) be a probabilistic metric space and \(f : G \to \Delta\) be a function. We say that \(f\) is continuous at \(x \in G\) if \(\lim_{t \to \infty} f(x_n)(t) = f(x)(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(f(x)\) and all sequence \((x_n)_n \subseteq G\) such that \(\lim_{n \to \infty} D(x_n, x) = H_0\) i.e. such that \(\lim_{n \to \infty} D(x_n, x)(t) = H_0(t)\) for all \(t \in \mathbb{R}\).

Now, we introduce the notion of probabilistic 1-Lipschitz map.

**Definition 2.2.** Let \((G, D, *)\) be a probabilistic metric space and \(f : G \to \Delta\) be a map. We say that \(f\) is a probabilistic 1-Lipschitz map if, for all \(x, y \in G\) we have:

\[
D(x, y) * f(y) \leq f(x).
\]

**Proposition 2.3.** Every probabilistic 1-Lipschitz map is continuous.
Proof. Let \( f \) be a probabilistic 1-Lipschitz map and \((x_n)_n \subset G\) be a sequence such that \( \lim_n D(x_n, x) = H_0 \). On one hand, since \((\Delta, \leq)\) is a complete lattice, the law \( \star \) is continuous and \( D(x_n, x) \star f(x) \leq f(x_n) \), then \( f(x) = H_0 \star f(x) = \lim_{n} D(x_n, x) \star f(x) \leq \lim_{n} f(x_n) \). Thus, \( f(x) \leq \lim_{n} f(x_n) \). On the other hand, since \( D(x_n, x) \star f(x_n) \leq f(x) \), it follows that \( \limsup_{n} f(x_n) = H_0 \star \limsup_{n} D(x_n, x) \star f(x_n) \leq f(x) \). Thus, \( \lim_{n} f(x_n) = f(x) \) and so \( f \) is continuous.

\[ \square \]

By \( \text{Lip}^{1}_1(G, \Delta) \) (resp. \( \text{Lip}^{1}_1(G, \Delta^+) \)), we denotes the space of all probabilistic 1-Lipschitz maps (resp. all \( \Delta^+ \)-valued 1-Lipschitz maps). For all \( x \in G \), by \( \delta_x : G \rightarrow \Delta^+ \) we denote the map \( \delta_x : y \mapsto D(y, x) \) and by \( \delta \), we denote the operator \( \delta : x \mapsto \delta_x \).

**Proposition 2.5.** Let \((G, D, \star)\) be a probabilistic metric space. Then, we have that \( \delta_a \in \text{Lip}^{1}_1(G, \Delta^+) \) for each \( a \in G \) and the map \( \delta : G \rightarrow \text{Lip}^{1}_1(G, \Delta^+) \) is injective.

**Proof.** The fact that \( \delta_a \in \text{Lip}^{1}_1(G, \Delta^+) \) for each \( a \in G \) follows from the property: \( D(x, y) \star D(y, a) \leq D(x, a) \) for all \( a, x, y \in G \). Now, let \( a, b \in G \) be such that \( \delta_a = \delta_b \). It follows that \( \delta_a(x) = \delta_b(x) \) for all \( x \in G \). In particular, for \( x = b \) we have that \( D(a, b) = \delta_a(b) = \delta_b(b) = H_0 \), which implies that \( a = b \).

Let \( f \in \text{Lip}^{1}_1(G, \Delta^+) \) and \( F \in \Delta^+ \), by \( \langle f, F \rangle : G \rightarrow \Delta^+ \), we denote the map defined by \( \langle f, F \rangle(x) := f(x) \star F \) for all \( x \in G \). We easily obtain the following proposition.

**Proposition 2.6.** Let \((G, D, \star)\) be a probabilistic metric space. Then, for all \( f \in \text{Lip}^{1}_1(G, \Delta^+) \) and all \( F \in \Delta^+ \), we have that \( \langle f, F \rangle \in \text{Lip}^{1}_1(G, \Delta^+) \).

Recall the Lipschitz extention result of Mac Shane in \[2\]: if \((X, d)\) is a metric space, \( A \) a nonempty subset of \( X \) and \( f : A \rightarrow \mathbb{R} \) is \( k \)-Lipschitz map, then there exist a \( k \)-Lipschitz map \( \bar{f} : X \rightarrow \mathbb{R} \) such that \( \bar{f}|_A = f \). We give bellow an analogous of this result for probabilistic 1-Lipschitz maps.

**Theorem 2.6.** Let \((G, D, \star)\) be a probabilistic metric space and \( A \) be a nonempty subset of \( G \). Let \( f : A \rightarrow \Delta \) be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map \( \bar{f} : G \rightarrow \Delta \) such that \( \bar{f}|_A = f \).

**Proof.** We define \( \bar{f} : G \rightarrow \Delta \) as follows: for all \( x \in G \),

\[ \bar{f}(x) := \sup_{a \in A} D(a, x) \star f(a). \]

We first prove that \( \bar{f}(x) = f(x) \) for all \( x \in A \). Indeed, let \( x \in A \). On one hand we have \( f(x) = H_0 \star f(x) = D(x, x) \star f(x) \leq \sup_{a \in A} D(a, x) \star f(a) = \bar{f}(x) \). On the other hand, since \( f \) is probabilistic 1-Lipschitz on \( A \) and \( x \in A \), then \( D(a, x) \star f(a) \leq f(x) \) for all \( a \in A \). It follows that \( f(x) := \sup_{a \in A} D(a, x) \star f(a) \leq f(x) \). Thus, \( \bar{f}(x) = f(x) \) for all \( x \in A \). Now, we show that \( \bar{f} \) is probabilistic 1-Lipschitz on \( G \). Indeed, let \( x, y \in G \). For all \( a \in A \) we have that \( D(a, x) \star D(x, y) \leq D(a, y) \). So,

\[ D(a, x) \star f(a) \star D(x, y) \leq D(a, y) \star f(a). \]

By taking the supremum over \( a \in A \) and using axiom \((vi)\) we get \( \bar{f}(x) \star D(x, y) \leq \bar{f}(y) \). Hence, \( \bar{f} \) is probabilistic 1-Lipschitz map on \( G \) that coincides with \( f \) on \( A \). \( \square \)
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