INTRODUCTION TO THE PROBABILISTIC 1-LIPSCHITZ MAPS

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Abstract. We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extenstion of real-valued Lipschitz functions from a subset of a metric space to the whole space.

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Contents

1. Introduction

We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane (see [2]) on the extenstion of real-valued Lipschitz functions from a subset of a metric space to the whole space.

A distribution function is a function $F : [-\infty, +\infty] \longrightarrow [0, 1]$ which is non-decreasing and left-continuous with $F(-\infty)) = 0$; $F(+\infty) = 1$. The set of all distribution functions will be denoted by Δ. The subset of Δ consisting on distributions F such that $F(0) = 0$ will be denoted by Δ^+. For $F, G \in \Delta^+$, the relation $F \leq G$ is meant by $F(t) \leq G(t)$, for all $t \in \mathbb{R}$. For all $a \in \mathbb{R}$, the distribution H_a is defined as follow

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a \\ 1, & \text{if } t > a \end{cases}$$

For $a = +\infty$,

$$H_\infty(t) = \begin{cases} 0, & \text{if } t \in [-\infty, +\infty[\\ 1, & \text{if } t = +\infty \end{cases}$$

It is well known that $(\Delta, \leq)$ and $(\Delta^+, \leq)$ are complet lattice with the minimal element H_∞ and the maximal element H_0 (see [1]). Thus, for any nonempty set I and any familly $(F_i)_{i\in I}$ of distributions in Δ (resp. in Δ^+), the function $F = \sup_{i\in I} F_i$ is also an element of Δ (resp. of Δ^+).

Axioms 1.1. *In this work, we assume that Δ^+ is equipped with a law $\star$ (a triangual function) satisfying the following axioms:*
(i) $F \star L \in \Delta^+$ for all $F, L \in \Delta^+$.
(ii) $F \star L = L \star F$ for all $F, L \in \Delta^+$.
(iii) $F \star (L \star K) = (F \star L) \star K$, for all $F, L, K \in \Delta^+$.
(iv) $F \star H_0 = F$ for all $F \in \Delta^+$.
(v) $F \leq L \Longrightarrow F \star K \leq L \star K$ for all $F, L, K \in \Delta^+$.
(vi) Let I be a set, $(F_i)_{i\in I}$ a familly of distributions in Δ^+ and $L \in \Delta^+$. Then, $\sup_{i\in I}(F_i \star L) = (\sup_{i\in I}(F_i)) \star L$.

Definition 1.2. *We say that $\star$ is continuous at $(F, L) \in \Delta^+ \times \Delta^+$ if $\lim_{n\to+\infty}(F_n \star L_n)(t) = (F \star L)(t)$ for all $t \in \mathbb{R}$ point of continuity of $F \star L$, whenever $\lim_{n\to+\infty} F_n(t) = F(t)$ for all $t \in \mathbb{R}$ point of continuity of F and $\lim_{n\to+\infty} L_n(t) = L(t)$ for all $t \in \mathbb{R}$ point of continuity of L.*

Example 1.3. *(see [3], [1]) Let $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be a left-continuous t-norm, then the operation $\star$ defined for all $F, L \in \Delta^+$ and for all $t \in \mathbb{R}$ by*

$$(F \star L)(t) := \sup_{s+u=t} T(F(s), L(u)) \tag{1.1}$$

is continuous at each point $(F, L) \in \Delta^+ \times \Delta^+$ and satisfies the axiomes (i)-(vi).

In all this work, we assume that Δ^+ is equipped with a continuous law $\star$ satisfying the axioms (i)-(vi).

Definition 1.4. *Let G be a set and let $D : G \times G \longrightarrow (\Delta^+, \star, \leq)$ be a map. We say that $(G, D, \star)$ is a probabilistic metric space if the following axioms (i)-(iii) hold:*
(i) $D(p, q) = H_0$ iff $p = q$.
(ii) $D(p, q) = D(q, p)$ for all $p, q \in G$
(iii) $D(p, q) \star D(q, r) \leq D(p, r)$ for all $p, q, r \in G$

2. Probabilistic 1-Lipschitz Maps

We define probabilistic continuous functions.

Definition 2.1. *Let $(G, D, \star)$ be a probabilistic metric space and $f : G \longrightarrow \Delta$ be a function. We say that f is continuous at $x \in G$ if $\lim_n f(x_n)(t) = f(x)(t)$ for all $t \in \mathbb{R}$ point of continuity of $f(x)$ and all sequence $(x_n)_n \subset G$ such that $\lim_n D(x_n, x) = H_0$ (i.e. such that $\lim_n D(x_n, x)(t) = H_0(t)$ for all $t \in \mathbb{R}$).*

Now, we introduce the notion of probabilistic 1-Lipschitz map.

Definition 2.2. *Let $(G, D, \star)$ be a probabilistic metric space and $f : G \longrightarrow \Delta$ be a map. We say that f is a probabilistic 1-Lipschitz map if, for all $x, y \in G$ we have:*

$$D(x, y) \star f(y) \leq f(x).$$

Proposition 2.3. *Every probabilistic 1-Lipschitz map is continuous.*

Proof. Let f be a probabilistic 1-Lipschitz map and $(x_n)_n \subset G$ be a sequence such that $\lim_n D(x_n, x) = H_0$. On one hand, since $(\Delta, \leq)$ is a complete lattice, the law $\star$ is continuous and $D(x_n, x) \star f(x) \leq f(x_n)$, then $f(x) = H_0 \star f(x) = \liminf_n D(x_n, x) \star f(x) \leq \liminf_n f(x_n)$. Thus, $f(x) \leq \liminf_n f(x_n)$. On the other hand, since $D(x_n, x) \star f(x_n) \leq f(x)$, it follows that $\limsup_n f(x_n) = H_0 \star \limsup_n f(x_n) = \limsup_n (D(x_n, x) \star f(x_n) \leq f(x)$. Thus, $\lim_n f(x_n) = f(x)$ and so f is continuous.

□

By $Lip^1_\star(G, \Delta)$ (resp. $Lip^1_\star(G, \Delta^+)$), we denotes the space of all probabilistic 1-Lipschitz maps (resp. all Δ^+-valued 1-Lipschitz maps). For all $x \in G$, by $\delta_x : G \longrightarrow \Delta^+$ we denote the map $\delta_x : y \mapsto D(y, x)$ and by δ, we denote the operator $\delta : x \mapsto \delta_x$.

Proposition 2.4. *Let $(G, D, \star)$ be a probabilistic metric space. Then, we have that $\delta_a \in Lip^1_\star(G, \Delta^+)$ for each $a \in G$ and the map $\delta : G \longrightarrow Lip^1_\star(G, \Delta^+)$ is injective.*

Proof. The fact that $\delta_a \in Lip^1_\star(G, \Delta^+)$ for each $a \in G$ follows from the property: $D(x, y) \star D(y, a) \leq D(x, a)$ for all $a, x, y \in G$. Now, let $a, b \in G$ be such that $\delta_a = \delta_b$. It follows that $\delta_a(x) = \delta_b(x)$ for all $x \in G$. In particular, for $x = b$ we have that $D(a, b) = \delta_a(b) = \delta_b(b) = H_0$, which implies that $a = b$. □

Let $f \in Lip^1_\star(G, \Delta^+)$ and $F \in \Delta^+$, by $\langle f, F\rangle : G \longrightarrow \Delta^+$, we denote the map defined by $\langle f, F\rangle(x) := f(x) \star F$ for all $x \in G$. We easly obtain the following propostion.

Proposition 2.5. *Let $(G, D, \star)$ be a probabilistic metric space. Then, for all $f \in Lip^1_\star(G, \Delta^+)$ and all $F \in \Delta^+$, we have that $\langle f, F\rangle \in Lip^1_\star(G, \Delta^+)$.*

Recall the Lipschitz extention result of Mac Shane in [2]: if (X, d) is a metric space, A a nonempty subset of X and $f : A \longrightarrow \mathbb{R}$ is k-Lipschitz map, then there exist a k-Lipschitz map $\tilde{f} : X \longrightarrow \mathbb{R}$ such that $\tilde{f}_{|A} = f$. We give bellow an analogous of this result for probabilistic 1-Lipschitz maps.

Theorem 2.6. *Let $(G, D, \star)$ be a probabilistic metric space and A be a nonempty subset of G. Let $f : A \longrightarrow \Delta$ be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map $\tilde{f} : G \longrightarrow \Delta$ such that $\tilde{f}_{|A} = f$.*

Proof. We define $\tilde{f} : G \longrightarrow \Delta$ as follows: for all $x \in G$,

$$\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a).$$

We first prove that $\tilde{f}(x) = f(x)$ for all $x \in A$. Indeed, let $x \in A$. On one hand we have $f(x) = H_0 \star f(x) = D(x, x) \star f(x) \leq \sup_{a \in A} D(a, x) \star f(a) = \tilde{f}(x)$. On the other hand, since f is probabilistic 1-Lipschitz on A and $x \in A$, then $D(a, x) \star f(a) \leq f(x)$ for all $a \in A$. It follows that $\tilde{f}(x) := \sup_{a \in A} D(a, x) \star f(a) \leq f(x)$. Thus, $\tilde{f}(x) = f(x)$ for all $x \in A$. Now, we show that $\tilde{f}$ is probabilistic 1-Lipschitz on G. Indeed, Let $x, y \in G$. For all $a \in A$ we have that $D(a, x) \star D(x, y) \leq D(a, y)$. So, $D(a, x) \star f(a) \star D(x, y) \leq D(a, y) \star f(a)$. By taking the supremum over $a \in A$ and using axiom (vi) we get $\tilde{f}(x) \star D(x, y) \leq f(y)$. Hence, $\tilde{f}$ is probabilistic 1-Lipschitz map on G that coincides with f on A. □

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