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Mohammed Bachir

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INTRODUCTION TO THE PROBABILISTIC 1-LIPSCHITZ MAPS

MOHAMMED BACHIR

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Laboratoire SAMM EA4543,
Université Paris 1 Panthéon-Sorbonne, centre P.M.F., France.

Abstract. We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

54E70, 47S50, 46S50

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1. Introduction

We introduce the notion of probabilistic 1-Lipschitz map defined on a probabilistic metric space and give an analogous to the result of Mac Shane on the extension of real-valued Lipschitz functions from a subset of a metric space to the whole space.

A distribution function is a function \( F : [-\infty, +\infty] \to [0,1] \) which is non-decreasing and left-continuous with \( F(-\infty) = 0 \); \( F(+\infty) = 1 \). The set of all distribution functions will be denoted by \( \Delta \). The subset of \( \Delta \) consisting on distributions \( F \) such that \( F(0) = 0 \) will be denoted by \( \Delta^+ \). For \( F, G \in \Delta^+ \), the relation \( F \leq G \) is meant by \( F(t) \leq G(t) \), for all \( t \in \mathbb{R} \). For all \( a \in \mathbb{R} \), the distribution \( H_a \) is defined as follow:

\[
H_a(t) = \begin{cases} 
0, & \text{if } t \leq a \\
1, & \text{if } t > a 
\end{cases}
\]

For \( a = +\infty \),

\[
H_\infty(t) = \begin{cases} 
0, & \text{if } t \in [-\infty, +\infty[ \\
1, & \text{if } t = +\infty 
\end{cases}
\]
It is well known that \((\Delta, \leq)\) and \((\Delta^+, \leq)\) are complete lattice with the minimal element \(H_\infty\) and the maximal element \(H_0\) (see [1]). Thus, for any nonempty set \(I\) and any family \((F_i)_{i \in I}\) of distributions in \(\Delta\) (resp. in \(\Delta^+\)), the function \(F = \sup_{i \in I} F_i\) is also an element of \(\Delta\) (resp. of \(\Delta^+\)).

**Axioms 1.1.** In this work, we assume that \(\Delta^+\) is equipped with a law \(*\) (a triangular function) satisfying the following axioms:

(i) \(F \ast L \in \Delta^+\) for all \(F, L \in \Delta^+\).

(ii) \(F \ast L = L \ast F\) for all \(F, L \in \Delta^+\).

(iii) \(F \ast (L \ast K) = (F \ast L) \ast K\), for all \(F, L, K \in \Delta^+\).

(iv) \(F \ast H_0 = F\) for all \(F \in \Delta^+\).

(v) \(F \leq L \implies F \ast K \leq L \ast K\) for all \(F, L, K \in \Delta^+\).

(vi) Let \(I\) be a set, \((F_i)_{i \in I}\) a family of distributions in \(\Delta^+\) and \(L \in \Delta^+\). Then, \(\sup_{i \in I} (F_i \ast L) = (\sup_{i \in I} F_i) \ast L\).

**Definition 1.2.** We say that \(*\) is continuous at \((F, L) \in \Delta^+ \times \Delta^+\) if \(\lim_{n \to +\infty} (F_n \ast L_n)(t) = (F \ast L)(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(F \ast L\), whenever \(\lim_{n \to +\infty} F_n(t) = F(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(F\) and \(\lim_{n \to +\infty} L_n(t) = L(t)\) for all \(t \in \mathbb{R}\) point of continuity of \(L\).

**Example 1.3.** (see [2], [3]) Let \(T : [0,1] \times [0,1] \to [0,1]\) be a left-continuous \(t\)-norm, then the operation \(*\) defined for all \(F, L \in \Delta^+\) and for all \(t \in \mathbb{R}\) by

\[
(F \ast L)(t) = \sup_{s+u=t} T(F(s), L(u))
\]

is continuous at each point \((F, L) \in \Delta^+ \times \Delta^+\) and satisfies the axioms (i)-(vi).

In all this work, we assume that \(\Delta^+\) is equipped with a continuous law \(*\) satisfying the axioms (i)-(vi).

**Definition 1.4.** Let \(G\) be a set and let \(D : G \times G \to (\Delta^+, \ast, \leq)\) be a map. We say that \((G, D, \ast)\) is a probabilistic metric space if the following axioms (i)-(iii) hold:

(i) \(D(p, q) = H_0\) iff \(p = q\).

(ii) \(D(p, q) = D(q, p)\) for all \(p, q \in G\)

(iii) \(D(p, q) \ast D(q, r) \leq D(p, r)\) for all \(p, q, r \in G\)

2. Probabilistic 1-Lipschitz Maps

We define probabilistic continuous functions.

**Definition 2.1.** Let \((G, D, \ast)\) be a probabilistic metric space and \(f : G \to \Delta\) be a function. We say that \(f\) is continuous at \(x \in G\) if \(\lim_{n \to \infty} D(x_n, x) = D(x, x)\) for all \(t \in \mathbb{R}\) point of continuity of \(f(x)\) and all sequence \((x_n)_n\) in \(G\) such that \(\lim_{n \to \infty} D(x_n, x) = H_0\) i.e. such that \(\lim_{n \to \infty} D(x_n, x)(t) = H_0(t)\) for all \(t \in \mathbb{R}\).

Now, we introduce the notion of probabilistic 1-Lipschitz map.

**Definition 2.2.** Let \((G, D, \ast)\) be a probabilistic metric space and \(f : G \to \Delta\) be a map. We say that \(f\) is a probabilistic 1-Lipschitz map if, for all \(x, y \in G\) we have:

\[
D(x, y) \ast f(y) \leq f(x).
\]

**Proposition 2.3.** Every probabilistic 1-Lipschitz map is continuous.
Proof. Let $f$ be a probabilistic 1-Lipschitz map and $(x_n)_n \subset G$ be a sequence such that $\lim_n D(x_n, x) = H_0$. On one hand, since $(\Delta, \leq)$ is a complete lattice, the law $*$ is continuous and $D(x_n, x) * f(x) \leq f(x_n)$, then $f(x) = H_0 * f(x) = \lim \inf_n D(x_n, x) * f(x) \leq \lim \inf_n f(x_n)$. Thus, $f(x) \leq \lim \inf_n f(x_n)$. On the other hand, since $D(x_n, x) * f(x_n) \leq f(x)$, it follows that $\lim \sup_n f(x_n) = H_0 * \lim \sup_n D(x_n, x) * f(x_n) \leq f(x)$. Thus, $\lim_n f(x_n) = f(x)$ and so $f$ is continuous.

By $\text{Lip}_1^1(G, \Delta)$ (resp. $\text{Lip}_1^1(G, \Delta^+)$), we denotes the space of all probabilistic 1-Lipschitz maps (resp. all $\Delta^+$-valued 1-Lipschitz maps). For all $x \in G$, by $\delta_x : G \rightarrow \Delta^+$ we denote the map $\delta_x : y \mapsto D(y, x)$ and by $\delta$, we denote the operator $\delta : x \mapsto \delta_x$.

**Proposition 2.4.** Let $(G, D, \ast)$ be a probabilistic metric space. Then, we have that $\delta_a \in \text{Lip}_1^1(G, \Delta^+)$ for each $a \in G$ and the map $\delta : G \rightarrow \text{Lip}_1^1(G, \Delta^+)$ is injective.

**Proof.** The fact that $\delta_a \in \text{Lip}_1^1(G, \Delta^+)$ for each $a \in G$ follows from the property: $D(x, y) * D(y, a) \leq D(x, a)$ for all $a, x, y \in G$. Now, let $a, b \in G$ be such that $\delta_a = \delta_b$. It follows that $\delta_a(x) = \delta_b(x)$ for all $x \in G$. In particular, for $x = b$ we have that $D(a, b) = \delta_a(b) = \delta_b(b) = H_0$, which implies that $a = b$. $\Box$

Let $f \in \text{Lip}_1^1(G, \Delta^+)$ and $F \in \Delta^+$, by $(f, F) : G \rightarrow \Delta^+$, we denote the map defined by $(f, F)(x) := f(x) * F$ for all $x \in G$. We easily obtain the following proposition.

**Proposition 2.5.** Let $(G, D, \ast)$ be a probabilistic metric space. Then, for all $f \in \text{Lip}_1^1(G, \Delta^+)$ and all $F \in \Delta^+$, we have that $(f, F) \in \text{Lip}_1^1(G, \Delta^+)$. 

Recall the Lipschitz extention result of Mac Shane in [2]: if $(X, d)$ is a metric space, $A$ a nonempty subset of $X$ and $f : A \rightarrow \mathbb{R}$ is $k$-Lipschitz map, then there exist a $k$-Lipschitz map $\tilde{f} : X \rightarrow \mathbb{R}$ such that $\tilde{f}|_A = f$. We give below an analogous of this result for probabilistic 1-Lipschitz maps.

**Theorem 2.6.** Let $(G, D, \ast)$ be a probabilistic metric space and $A$ be a nonempty subset of $G$. Let $f : A \rightarrow \Delta$ be a probabilistic 1-Lipschitz map. Then, there exists a probabilistic 1-Lipschitz map $\tilde{f} : G \rightarrow \Delta$ such that $\tilde{f}|_A = f$.

**Proof.** We define $\tilde{f} : G \rightarrow \Delta$ as follows: for all $x \in G$,

$$\tilde{f}(x) := \sup_{a \in A} D(a, x) * f(a).$$

We first prove that $\tilde{f}(x) = f(x)$ for all $x \in A$. Indeed, let $x \in A$. On one hand we have $f(x) = H_0 * f(x) = D(x, x) * f(x) \leq \sup_{a \in A} D(a, x) * f(a) = \tilde{f}(x)$. On the other hand, since $f$ is probabilistic 1-Lipschitz on $A$ and $x \in A$, then $D(a, x) * f(a) \leq f(x)$ for all $a \in A$. It follows that $\tilde{f}(x) := \sup_{a \in A} D(a, x) * f(a) \leq f(x)$. Thus, $\tilde{f}(x) = f(x)$ for all $x \in A$. Now, we show that $\tilde{f}$ is probabilistic 1-Lipschitz on $G$. Indeed, let $x, y \in G$. For all $a \in A$ we have that $D(a, x) * D(x, y) \leq D(a, y)$. So, $D(a, x) * f(a) * D(x, y) \leq D(a, y) * f(a)$. By taking the supremum over $a \in A$ and using axiom (vi) we get $\tilde{f}(x) * D(x, y) \leq f(y)$. Hence, $\tilde{f}$ is probabilistic 1-Lipschitz map on $G$ that coincides with $f$ on $A$. $\Box$
References

