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# FINITELY DETERMINED FUNCTIONS AND CONVEX OPTIMIZATION.

M. BACHIR, A. FABRE AND S. TAPIA-GARCÍA

ABSTRACT. We study the notion of *finitely determined functions* defined on a topological vector space  $E$  equipped with a biorthogonal system. This notion will be used to obtain a necessary and sufficient condition for a convex function to attain a minimum at some point. An application to the Karush-Kuhn-Tucker theorem will be given. For real-valued convex functions defined on a Banach space with a Schauder basis, the notion of finitely determined function coincides with the classical continuity but outside the convex case there are many finitely determined nowhere continuous functions.

**Keyword, phrase:** Convex optimization, Schauder basis, Finitely determined function, Directional derivatives, Karush-Kuhn-Tucker theorem.

**msc:** 46N10, 58B10, 49J50.

## CONTENTS

1. Introduction	1
2. Finitely determined functions	4
2.1. Banach space with a Schauder basis	4
2.2. Topological vector space without Schauder basis	8
3. Necessary and Sufficient Condition of Convex Optimality	11
4. Application to the Karush-Kuhn-Tucker theorem	15
5. Examples	17
References	18

## 1. INTRODUCTION

Let  $E$  be a topological vector space over the field  $\mathbb{R}$  and  $E^*$  its topological dual. Let  $(e_n)$  be a linearly independent family of elements of  $E$  and  $(e_n^*)$  be a family of elements of  $E^*$ . The pair  $(e_n, e_n^*)$  is said to be a biorthogonal system if  $\langle e_n^*, e_n \rangle = 1$  for all  $n \in \mathbb{N}$  and  $\langle e_n^*, e_k \rangle = 0$  if  $n \neq k$ . Furthermore,  $(e_n, e_n^*)$

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it is called fundamental if  $E = \overline{\text{span}}(e_n : n \in \mathbb{N})$ . The linear mappings  $P_k : E \rightarrow E$  are defined for all  $k \in \mathbb{N}$  as follows

$$x \xrightarrow{P_k} P_k(x) = \sum_{n=0}^k \langle e_n^*, x \rangle e_n.$$

A well known result asserts that each Banach space  $E$  contains a biorthogonal system  $(e_n, e_n^*)_n$ . Moreover, whenever  $E$  is separable there exists a fundamental biorthogonal system, see [7].

Throughout the manuscript, we assume that the topological vector space  $E$  is equipped with a biorthogonal system  $(e_n, e_n^*)$ . For further information about biorthogonal systems, we refer to [1], [9], [10], [5] and [13].

**Definition 1.** We say that  $f : E \rightarrow \mathbb{R}$  is finitely determined by the biorthogonal system  $(e_n, e_n^*)$  with respect to  $a \in E$  if we have:

$$f(x) = \lim_{n \rightarrow +\infty} f(a + P_n(x - a)), \quad \forall x \in E,$$

If the above equality is satisfied with respect to all  $a \in E$ , then we say that  $f$  is finitely determined by the biorthogonal system  $(e_n, e_n^*)$ . We say that  $f$  is inf-finitely determined by the biorthogonal system  $(e_n, e_n^*)$  with respect to  $a \in E$  if we have:

$$f(x) \geq \inf_{n \in \mathbb{N}} f(a + P_n(x - a)), \quad \forall x \in E,$$

If the above inequality is satisfied with respect to all  $a \in E$ , then we say that  $f$  is inf-finitely determined.

When there is no confusion with the related biorthogonal system, we will simply say that  $f$  is finitely determined (resp. inf-finitely determined) at  $a$  or with respect to  $a$ .

Clearly, every finitely determined function is inf-finitely determined. The aim of this paper is to study the notions of finitely determined and inf-finitely determined function and their applications to optimization. The motivation behind the study of these notions, lies in the following simple observations:

(1) If  $f$  is inf-finitely determined (not necessarily convex), then  $f$  has a global minimum at  $\bar{x} = 0$  (we take  $\bar{x} = 0$  for simplicity) if and only if the restriction  $f|_{P_k(E)}$  has a global minimum at 0 for each  $k \in \mathbb{N}$ , where  $P_k(E)$  denotes the image of  $E$  under the linear mapping  $P_k$ . Since  $P_k(E)$  is of finite dimension for each  $k \in \mathbb{N}$ , the terminology of *finitely determined function* is motivated.

(2) Let  $E = l^\infty(\mathbb{N})$  the Banach space of bounded sequences. We consider the biorthogonal system given by the canonical basis  $(e_n)$  of  $c_0(\mathbb{N})$ , seen as a subspace of  $E$ , and the coordinate functionals  $(e_n^*)$ . Let  $p : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$

defined by  $p(x) = \limsup_n |x_n|$ . We know from [14, Example 1.21] that  $p$  is nowhere Gâteaux-differentiable. On the other hand, clearly,  $p$  is convex and inf-finitely determined with respect to each point  $a$  of  $c_0(\mathbb{N})$  ( $p$  is also a norm continuous seminorm). Moreover, the directional derivative with respect to the canonical basis  $(e_n)$  of  $l^\infty(\mathbb{N})$ , exists and is equal to 0 at each point of  $l^\infty(\mathbb{N})$  (see Example 1), that is, for all  $x \in l^\infty(\mathbb{N})$  and all  $n \in \mathbb{N}$ ,

$$p'(x; e_n) := \lim_{\substack{t \rightarrow 0 \\ t \neq 0}} \frac{1}{t} (p(x + te_n) - p(x)) = 0 \quad (\star)$$

We then notice trivially, that  $p$  has a minimum at a point  $a$  iff  $a \in c_0(\mathbb{N})$ . Also,  $p$  is inf-finitely determined with respect  $a$  iff  $a \in c_0(\mathbb{N})$ . The question in this paper is: is it true that every convex function  $f : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$  which is inf-finitely determined with respect to some point  $x_0 \in l^\infty(\mathbb{N})$  and satisfies the equation  $(\star)$  at  $x_0$  (weaker than Gâteaux-differentiability), has necessarily a minimum at this point? We answer this question positively, even in a more general framework (Theorem 3). Thus, for inf-finitely determined function with respect to  $x$ , the criterion  $(\star)$  is sufficient to characterize a minimum at  $x$ , and the stronger assumption of Gâteaux-differentiability can be relaxed.

We say that  $(e_n)$  is a topological basis (or Schauder basis) of  $E$  if for each  $x \in E$ , there exists a unique sequence  $(a_n)$  of real number such that  $x = \sum_{n=0}^{+\infty} a_n e_n = \lim_n \sum_{i=1}^n a_i e_i$ , where the convergence is understood with respect to the topology of  $E$ . In this case we have  $a_n = \langle e_n^*, x \rangle$  for all  $n \in \mathbb{N}$ . If  $E$  is a topological vector space with a topological basis  $(e_n)$ , we have  $a + P_n(x - a) \rightarrow x$ , for all  $a, x \in E$ . In this case, every continuous function is finitely determined by  $(e_n, e_n^*)$  and we prove in Corollary 4, that the existence of partial derivatives (in the directions  $(e_n)$ , when it is a Schauder basis) coincides with Gateaux-differentiability for convex continuous functions.

The space  $(FD_b(E), \|\cdot\|_\infty)$  (resp.  $(C_b(E), \|\cdot\|_\infty)$ ) denotes the set of all real-valued bounded finitely determined functions on  $E$  with respect to a given biorthogonal system  $(e_n, e_n^*)$  (resp. of all real-valued bounded continuous functions on  $E$ ) equipped with the sup-norm. It is easy to see that  $(FD_b(E), \|\cdot\|_\infty)$  is a Banach algebra.

This paper is organized as follows. In Section 2, we prove that if  $E$  is a Banach space and  $(e_n)_n$  is a Schauder basis, the property of finite determination coincides with continuity for real-valued convex functions. Nonetheless, outside the convex case there are a many finitely determined nowhere continuous functions, in particular we prove that  $FD_b(E) \setminus C_b(E)$  is an open dense subset of  $FD_b(E)$  (Theorem 1 and Theorem 2). On the other hand, if  $E$  is a separable Banach space without Schauder basis and  $(e_n, e_n^*)$  is a

fundamental biorthogonal system of  $E$ , then the norm of  $E$  is not finitely determined by the biorthogonal system  $(e_n, e_n^*)$  (Corollary 1). A characterization of Schauder basis in term of the equivalence between continuity and the notion of finitely determined for convex functions is given in Corollary 2. In Section 3, using the notion of inf-finitely determined functions, we give a necessary and sufficient condition for a convex function to have a minimum at some point (Theorem 3). In Section 4, we use this result to generalize the Karush-Kuhn-Tucker theorem. Finally, in Section 5, we give some examples.

**Notation:** Throughout this paper,  $E$  denotes a topological vector (or Banach) space,  $(x_n) \subset E$  denotes a sequence in  $E$  and whenever  $E$  is a normed space,  $B_E(x, \rho)$  denotes the open ball centered at  $x$  of radius  $\rho$ .  $E^*$  denotes the dual space of  $E$  and  $\langle \cdot, \cdot \rangle$  the duality product. For a convex function  $f$ ,  $\partial f(x)$  denotes the convex subdifferential of  $f$  at the point  $x$ .

## 2. FINITELY DETERMINED FUNCTIONS

In this section, we study some properties of inf-finitely determined and finitely determined functions in Banach spaces equipped or not with a Schauder basis.

**2.1. Banach space with a Schauder basis.** Recall that a Banach space  $(E, \|\cdot\|)$  with a Schauder basis is necessarily separable. In our proofs, we will use the following well known result.

**Lemma 1.** [1, Proposition 1.1.9] *A sequence  $(e_n)$  of nonzero vectors of a Banach space  $E$  is a Schauder basis in  $\overline{\text{span}}(e_n : n \in \mathbb{N})$  if and only if there is a positive constant  $K$  such that*

$$\left\| \sum_{k=1}^m a_k e_k \right\| \leq K \left\| \sum_{k=1}^n a_k e_k \right\|$$

for every sequence of scalars  $(a_k)$  and all integers  $m, n$  such that  $m \leq n$ .

*Remark 1.* In the context of the paper, Lemma 1 can be reformulated as follows: A sequence  $(e_n) \subset E$  is a Schauder basis if and only if the linear operators  $(P_n)$  are uniformly bounded, i.e. there exists some  $K > 0$  such that  $\|P_n\| < K$  for all  $n \in \mathbb{N}$ .

The following is the main theorem of this subsection:

**Theorem 1.** *Let  $E$  be a Banach space equipped with a fundamental biorthogonal system  $(e_n, e_n^*)$  such that  $(e_n)$  is a Schauder basis. Then, for every real-valued convex function, the notions of finitely determined by  $(e_n, e_n^*)$  and classical continuity coincides.*

Before proceeding with the proof of Theorem 1, we recall simple facts about convex functions: every convex function  $f : Y \rightarrow \mathbb{R}$  defined on a finite dimensional Banach space  $Y$  is continuous and every convex function  $F : E \rightarrow \mathbb{R}$  defined on a Banach space  $E$  is continuous if and only if it is locally bounded from above at each point  $x \in E$ . Moreover, the last property is equivalent to be locally bounded from above at just one point  $x \in E$ . Also, we need the following lemma (see [4, Theorem 5.43, p. 188]).

**Lemma 2.** *Let  $f : E \rightarrow \mathbb{R}$  be a convex function and  $Y, Z$  be two closed subspace of  $E$  such that  $E = Y + Z$  and  $Y \cap Z = 0$ . Let  $x_0 = y_0 + z_0 \in E$ , where  $y_0 \in Y$  and  $z_0 \in Z$  fixed. Let  $g : Y \rightarrow \mathbb{R}, h : Z \rightarrow \mathbb{R}$  be the convex functions defined by  $g(y) = f(y + z_0)$  and  $h(z) = f(y_0 + z)$ . Then  $f$  is continuous if and only if both  $g$  and  $h$  are continuous.*

*Proof.* The necessity is straightforward. To prove the sufficiency, we show that  $f$  is locally bounded. Let  $\bar{x} = \bar{y} + \bar{z} \in E$  where  $\bar{y} \in Y$  and  $\bar{z} \in Z$ . Let  $\epsilon > 0, u \in B_Y(\bar{y}, \epsilon)$  and  $v \in B_Z(\bar{z}, \epsilon)$ , then:

$$\begin{aligned} f(u + v) &= f\left(u - \frac{y_0}{2} + \frac{z_0}{2} + v - \frac{z_0}{2} + \frac{y_0}{2}\right) \\ &\leq \frac{1}{2}f(2u - y_0 + z_0) + \frac{1}{2}f(2v - z_0 + y_0) \\ &= \frac{1}{2}g(2u - y_0) + \frac{1}{2}h(2v - z_0). \end{aligned}$$

Since  $g$  and  $h$  are continuous, then  $f$  is locally bounded.  $\square$

We also highlight the definition of a finitely determined function  $f$  with respect to the point  $a = 0$ , i.e. if  $f$  is finitely determined at 0, it holds:

$$(1) \quad f(x) = \lim_n f(P_n(x)) \text{ for all } x \in E.$$

*Proof of Theorem 1.* Let  $f : E \rightarrow \mathbb{R}$  be a continuous function. Since  $(e_n)_n$  is a Schauder basis of  $E$ ,  $f$  is finitely determined. Indeed, for every  $a \in E$  and  $x \in E$ , the sequence  $(a + P_n(x - a))_n$  converges to  $x$ . On the other hand, let us assume by contradiction that there exists a finitely determined convex function  $f : E \rightarrow \mathbb{R}$  which is discontinuous. The idea of the proof is to find a point  $\bar{x} \in E$  (by induction) such that  $f(\bar{x})$  must take the value  $+\infty$ . Since  $f$  is convex and its domain is  $E$ , it must not be locally bounded from above at any point, in particular at 0. Let  $x_1 \in E$  such that  $\|x_1\| < 1$  and  $f(x_1) > 1$ . By equation (1), we get  $N_1 \in \mathbb{N}$  such that  $f(P_{N_1}(x)) > 1$ . Let us call  $\bar{x}_1 = P_{N_1}(x_1)$ . In order to use Lemma 2, consider the subspaces  $Y_1 := \text{span}(e_n : n \leq N_1)$  and  $Z_1 := \overline{\text{span}}(e_n : n > N_1)$  and the point  $\bar{x}_1 = \bar{x}_1 + 0$ . Since  $f$  is discontinuous and  $Y_1$  is a finite dimensional space, then the function  $g_1 : Z_1 \rightarrow \mathbb{R}$  defined by  $g_1(z) := f(\bar{x}_1 + z)$  is also discontinuous and satisfies equation (1). Inductively, suppose that we have constructed the vectors  $\{\bar{x}_i\}_{i=1}^k$  and the family  $\{\bar{N}_i\}_{i=1}^k$ ,

where  $\bar{x}_i \in \text{span}(e_n : n \in \{N_{i-1} + 1, \dots, N_i\})$ ,  $N_0 = -1$ ,  $g_{i-1}(\bar{x}_i) > i$  and  $\|\bar{x}_i\| \leq \|P_{N_i}\|/2^{i-1}$ . Let us define  $Z_k = \overline{\text{span}}(e_n : n > N_k)$ . By Lemma 2, the convex function  $g_k : Z_k \rightarrow \mathbb{R}$  defined by  $g_k(z) = f(\sum_{i=1}^k \bar{x}_i + z)$  is discontinuous and equation (1) is still valid for it. Since  $g_k$  is not locally bounded from above at 0, there exists  $x_{k+1} \in Z_k$  such that  $\|x_{k+1}\| \leq 1/2^k$  and  $g_k(x_{k+1}) > k + 1$ . But using equation (1), we get an integer  $N_{k+1} > N_k$  such that the vector  $\bar{x}_{k+1} = P_{N_{k+1}}(x_{k+1})$  also satisfies  $g_k(\bar{x}_{k+1}) > k + 1$ . Having constructed a sequence  $(\bar{x}_n) \subset E$  using the previous induction, we can check that the function  $f$  at the point:

$$\bar{x} = \sum_{i=1}^{\infty} \bar{x}_i,$$

must take the value  $+\infty$ . In fact, to show that the point  $\bar{x}$  is well defined, we just use Lemma 1 to recall that the norm of the projections  $(P_n)_n$  are uniformly bounded and compute:

$$\sum_{k=1}^{\infty} \|\bar{x}_k\| = \sum_{k=1}^{\infty} \|P_{N_k}(x_k)\| \leq \sum_{k=1}^{\infty} \frac{\sup_k \{\|P_k\|\}}{2^k} < \infty.$$

Finally, using equation (1) we deduce that:

$$f(\bar{x}) = \lim_k f(P_k(\bar{x})) = \lim_k f(P_{N_k}(\bar{x})) = \lim_k g_k(\bar{x}_{k+1}) = \infty,$$

which leads to a contradiction. Hence, we proved that, for real-valued convex functions, the notions of finitely determined and classical continuity coincide.  $\square$

In the following theorem, we prove that outside the convex case there are lots of finitely determined nowhere continuous functions.

**Theorem 2.** *Under the hypothesis of Theorem 1, there exists a  $G_\delta$  dense subset  $G$  of  $E$  such that for every  $f \in C_b(E)$  and every  $\varepsilon > 0$ , there exists  $\tilde{f}_\varepsilon \in FD_b(E)$  nowhere continuous on  $E \setminus f^{-1}(0)$  such that  $\|\tilde{f}_\varepsilon - f\|_\infty < \varepsilon$  and  $\tilde{f}_\varepsilon = f$  on  $f^{-1}(0) \cup G$ . In particular,  $C_b(E)$  is a closed subspace of  $(FD_b(E), \|\cdot\|_\infty)$  with empty interior.*

*Proof.* Let us define the following function:

$$\sigma(t) = \begin{cases} 1 & \text{if } t \in \mathbb{Q} \\ 0 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Let  $\varepsilon > 0$  and  $f \in C_b(E) \setminus \{0\}$ . Let us set

$$\tilde{f}_\varepsilon(x) = \left(1 - \frac{\varepsilon \sigma(\langle e_0^*, x \rangle)}{\|f\|_\infty}\right) f(x) \text{ for all } x \in E.$$

Then, we have that for all  $x \in E$

$$\begin{aligned} |\tilde{f}_\varepsilon(x) - f(x)| &= \frac{\varepsilon \sigma(\langle e_0^*, x \rangle)}{\|f\|_\infty} |f(x)| \\ &\leq \varepsilon. \end{aligned}$$

It follows that  $\|\tilde{f}_\varepsilon - f\|_\infty \leq \varepsilon$  and  $\tilde{f}_\varepsilon = f$  on  $f^{-1}(0) \cup (e_0^*)^{-1}(\mathbb{R} \setminus \mathbb{Q})$ . Let us set  $\mathbb{Q} = \cup_{n \in \mathbb{N}} \{q_n\}$ , we have

$$G := (e_0^*)^{-1}(\mathbb{R} \setminus \mathbb{Q}) = \cap_{n \in \mathbb{N}} (e_0^*)^{-1}(\mathbb{R} \setminus \{q_n\}).$$

For each  $n \in \mathbb{N}$ , the set  $(e_0^*)^{-1}(\mathbb{R} \setminus \{q_n\})$  is an open dense subset of  $E$  (in fact the complement of an affine subspace). Thus  $G$  is a  $G_\delta$  dense subset of  $E$  and  $\tilde{f}_\varepsilon = f$  on  $f^{-1}(0) \cup G$ . To see that  $\tilde{f}_\varepsilon \in FD_b(E)$ , it suffices to show that the function  $\tilde{\sigma} : x \mapsto \sigma(\langle e_0^*, x \rangle)$  belongs to  $FD_b(E)$ , since  $FD_b(E)$  is a Banach algebra and  $C_b(E) \subset FD_b(E)$ . Indeed, for all  $a, x \in E$ , we have  $\tilde{\sigma}(a + P_k(x - a)) = \tilde{\sigma}(x)$ . Thus,  $\tilde{\sigma} \in FD_b(E)$  and so we have  $\tilde{f}_\varepsilon \in FD_b(E)$ . Now, we prove that  $\tilde{f}_\varepsilon$  is nowhere continuous on  $E \setminus f^{-1}(0)$ . Indeed, let  $x \in E$  and  $a \in E \setminus f^{-1}(0)$ ,

$$\begin{aligned} |\tilde{f}_\varepsilon(x) - \tilde{f}_\varepsilon(a)| &= \left| \left(1 - \frac{\varepsilon \sigma(\langle e_0^*, x \rangle)}{\|f\|_\infty}\right) (f(x) - f(a)) + \frac{\varepsilon (\sigma(\langle e_0^*, a \rangle) - \sigma(\langle e_0^*, x \rangle))}{\|f\|_\infty} f(a) \right| \\ (2) \qquad \qquad \qquad &\geq \left| \frac{\varepsilon (\sigma(\langle e_0^*, a \rangle) - \sigma(\langle e_0^*, x \rangle))}{\|f\|_\infty} f(a) \right| - \left| 1 - \frac{\varepsilon \sigma(\langle e_0^*, x \rangle)}{\|f\|_\infty} \right| |f(x) - f(a)| \end{aligned}$$

*Case 1:* if  $\langle e_0^*, a \rangle \in \mathbb{R} \setminus \mathbb{Q}$ , we choose rational numbers  $r_k(a) \in \mathbb{Q}$  such that  $|r_k(a) - \langle e_0^*, a \rangle| \leq 2^{-k}$  for all  $k \in \mathbb{N}$ , and we set  $x_k = a + (r_k(a) - \langle e_0^*, a \rangle)e_0$  for all  $k \in \mathbb{N}$ . Then,  $\|x_k - a\| \leq 2^{-k}$  for all  $k \in \mathbb{N}$  and  $|\sigma(\langle e_0^*, a \rangle) - \sigma(\langle e_0^*, x_k \rangle)| = 1$ . It follows from (2) that

$$|\tilde{f}_\varepsilon(x_k) - \tilde{f}_\varepsilon(a)| \geq \frac{\varepsilon |f(a)|}{\|f\|_\infty} - \left| 1 - \frac{\varepsilon \sigma(\langle e_0^*, x_k \rangle)}{\|f\|_\infty} \right| |f(x_k) - f(a)|.$$

Since  $f$  is continuous, sending  $k$  to  $+\infty$ , we have that

$$\liminf_k |\tilde{f}_\varepsilon(x_k) - \tilde{f}_\varepsilon(a)| \geq \frac{\varepsilon |f(a)|}{\|f\|_\infty} > 0,$$

which implies that  $\tilde{f}_\varepsilon$  is not continuous at  $a$ .

*Case 2:* in a similar way, if  $\langle e_0^*, a \rangle \in \mathbb{Q}$ , we choose irrational numbers  $r_k(a) \in \mathbb{R} \setminus \mathbb{Q}$  such that  $|r_k(a) - \langle e_0^*, a \rangle| \leq 2^{-k}$  for all  $k \in \mathbb{N}$ , and we put  $x_k = a + (r_k(a) - \langle e_0^*, a \rangle)e_0$  for all  $k \in \mathbb{N}$ . Then,  $\|x_k - a\| \leq 2^{-k}$  for all  $k \in \mathbb{N}$



and  $|\sigma(\langle e_0^*, a \rangle) - \sigma(\langle e_0^*, x_k \rangle)| = 1$ . Then, using (2) and sending  $k$  to  $+\infty$ , we have that

$$\liminf_k |\tilde{f}_\varepsilon(x_k) - \tilde{f}_\varepsilon(a)| \geq \frac{\varepsilon |f(a)|}{\|f\|_\infty} > 0,$$

which implies also that  $\tilde{f}_\varepsilon$  is not continuous at  $a$ .

Finally, we proved that for every  $f \in C_b(E) \setminus \{0\}$  and every  $\varepsilon > 0$ , there exists  $\tilde{f}_\varepsilon \in FD_b(E)$  nowhere continuous on  $E \setminus f^{-1}\{0\}$  such that  $\|\tilde{f}_\varepsilon - f\|_\infty \leq \varepsilon$  and  $\tilde{f}_\varepsilon = f$  on  $f^{-1}\{0\} \cup G$  (the case of  $f = 0$  is also clear). Since  $C_b(E)$  is a closed subset of  $FD_b(E)$ , it is now clear that  $FD_b(E) \setminus C_b(E)$  is an open and dense subset of  $FD_b(E)$ . This concludes the proof.  $\square$

**2.2. Topological vector space without Schauder basis.** In the following proposition, it is shown that a convex norm-continuous function is not necessarily finitely determined by a biorthogonal system if the sequence is not a Schauder basis. Note that, finding a separable Banach space without a Schauder basis is non-trivial result due to P. Enflo [8].

**Proposition 1.** *Let  $E$  be a separable Banach space without Schauder basis. Then for every fundamental biorthogonal system  $(e_n, e_n^*)$  there exists a continuous linear form  $y^* \in E^*$  which is not finitely determined by  $(e_n, e_n^*)$ .*

*Proof.* First, since the function  $\phi_n : E^* \rightarrow \mathbb{R}$  defined by  $\phi_n(x^*) := \|x^* \circ P_n\|$  is convex continuous for each  $n \in \mathbb{N}$ , the function  $\Phi : E^* \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by  $\Phi(x^*) := \sup_n \phi_n$  is convex and lower semi continuous (lsc from now on). We prove that there exists  $x^* \in E^*$  such that  $\Phi(x^*) = +\infty$ . Indeed, by Lemma 1 we already know that  $\sup_n \|P_n\| = +\infty$ , with which we can compute the following estimation:

$$\begin{aligned} +\infty &= \sup_n \|P_n\| = \sup_n \sup_{\|x\|=1} \|P_n x\| = \sup_n \sup_{\|x\|=1} \sup_{\|x^*\|=1} |\langle x^*, P_n x \rangle| \\ &= \sup_n \sup_{\|x^*\|=1} \sup_{\|x\|=1} |\langle x^* \circ P_n, x \rangle| \leq \sup_n \sup_{\|x^*\|=1} \|x^* \circ P_n\| = \sup_{\|x^*\|=1} \Phi(x^*), \end{aligned}$$

thus, we can deduce that the function  $\Phi$  is not locally bounded at 0, then it is not continuous. Since  $\Phi$  is lsc, there exists some  $\bar{x}^* \in E^*$  such that  $\Phi(\bar{x}^*) = +\infty$ . Hence,  $\bar{x}^*$  is not finitely determined by  $(e_n, e_n^*)$ , otherwise, from equation (1) and Banach-Steinhaus Theorem we have that  $\Phi(\bar{x}^*) \in \mathbb{R}$ , which is a contradiction.  $\square$

*Remark 2.* Since  $\Phi$  is a convex lsc function, in the proof of Proposition 1 we have proven that its domain has empty interior.

By  $f^*$  we denote the Fenchel conjugate of a function  $f$ : for all  $x^* \in E^*$

$$f^*(x^*) := \sup_{x \in E} \{\langle x^*, x \rangle - f(x)\}.$$

**Corollary 1.** *Let  $E$  be a separable Banach space without Schauder basis and let  $(e_n, e_n^*)_n$  be a fundamental biorthogonal system of  $E$ . Then, there is no finitely determined (by  $(e_n, e_n^*)$ ) convex continuous function  $f : E \rightarrow \mathbb{R}$  such that  $\text{int}(\text{dom}(f^*)) \neq \emptyset$ . In particular, the norm  $\|\cdot\|$  of  $E$  is never finitely determined.*

*Proof.* Let  $f : E \rightarrow \mathbb{R}$  be a convex function such that the domain of its Fenchel conjugate  $f^*$  has nonempty interior. Let  $x^* \in \text{int}(\text{dom}(f^*))$ . Since  $f^*$  is convex and lsc, it is bounded from above at  $x^*$ . Then there exists some  $\rho > 0$  and  $M > 0$  such that  $f^*(y^*) \leq M$  for all  $y^* \in B_{E^*}(x^*, \rho)$ . By definition of  $f^*$ , we have:

$$(3) \quad \langle y^*, x \rangle \leq f^*(y^*) + f(x) \leq M + f(x), \quad \forall x \in E, \quad \forall y^* \in B_{E^*}(x^*, \rho).$$

Let  $\Phi : E^* \rightarrow \mathbb{R} \cup \{+\infty\}$  be the function defined in the proof of Proposition 1. We know that its domain has empty interior, thus there exists a  $\bar{y}^* \in B_{E^*}(x^*, \rho)$  such that  $\Phi(\bar{y}^*) = +\infty$ . By Banach-Steinhaus Theorem, there exists  $x \in X$  such that the sequence  $(\bar{y}^* \circ P_n(x))$  is not bounded from above. Hence, by equation (3) we have that equation (1) is not satisfied at the point  $x$ . Finally, since the Fenchel conjugate of the norm is the function  $f : E^* \rightarrow \mathbb{R} \cup \{\infty\}$  such that  $f(y^*) = 0$  if  $\|y^*\| \leq 1$  and  $+\infty$  otherwise, we conclude the theorem.  $\square$

The notion of finitely determined function, applied to convex functions, characterizes the Banach spaces with a Schauder basis.

**Corollary 2.** *Let  $E$  be a separable Banach space and let  $(e_n, e_n^*)_n$  be a fundamental biorthogonal system of  $E$ . Then,  $(e_n)$  is a Schauder basis if and only if the notions of finitely determined by  $(e_n, e_n^*)$  and norm continuity coincide for real-valued convex functions.*

*Proof.* The proof is a consequence of Theorem 1 and Corollary 1.  $\square$

We give below a useful example of inf-finitely determined (with respect to some points) convex function having directional derivative but which is nowhere Gâteaux-differentiable. We need the following definition.

**Definition 2.** (*Directional-differentiability*) *Let  $E$  be a Banach space and let  $(e_n, e_n^*)_n$  be a biorthogonal system of  $E$ . We say that  $f$  is differentiable at  $a$  in the directions  $(e_n)$  if the following limit exists for all  $n \in \mathbb{N}$*

$$f'(a; e_n) := \lim_{\substack{t \rightarrow 0 \\ t \neq 0}} \frac{1}{t} \left( f(a + te_n) - f(a) \right).$$

*Example 1.* Let  $E = l^\infty(\mathbb{N})$  the Banach space of bounded sequences. We denote  $e_n := (\delta_j^n)$  the elements of  $l^\infty(\mathbb{N})$  where  $\delta_j^n$  is the Kronecker symbol satisfying  $\delta_j^n = 1$  if  $j = n$  and 0 if  $j \neq n$ . Let  $(e_n, e_n^*)$  be the natural

biorthogonal system of  $l^\infty(\mathbb{N})$ . Let  $p : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$  be the function defined for all  $x = (x_n) \in l^\infty(\mathbb{N})$  by

$$p(x) = \limsup |x_n|.$$

Then,

(1)  $p$  is a continuous seminorm ( $p(x) \leq \|x\|_\infty$  for all  $x \in l^\infty(\mathbb{N})$ ), is differentiable in the directions  $(e_n)_{n \geq 1}$  at each  $x \in l^\infty(\mathbb{N})$  and we have  $p'(x; e_n) = 0$  for all  $n \in \mathbb{N}^*$  and all  $x \in l^\infty(\mathbb{N})$ . However,  $p$  is nowhere Gâteaux-differentiable.

(2)  $p$  is inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to  $a$  if and only if  $p(a) = 0$  (i.e.  $a \in c_0(\mathbb{N})$ ).

*Proof.* It is well known that  $p$  is a continuous seminorm (with respect to the norm  $\|\cdot\|_\infty$ ), nowhere Gâteaux-differentiable (see [14, Example 1.21]). We show that  $p$  is differentiable at each  $x$  in the directions  $(e_n)$ . Indeed, for each fixed integer  $n \in \mathbb{N}^*$  and each  $t \in \mathbb{R}$ , it is easy to see that  $p(x + te_n) = p(x)$ . It follows that  $p'(x; e_n) = 0$  for all  $n \in \mathbb{N}^*$  and all  $x \in l^\infty(\mathbb{N})$ . On the other hand,  $p$  is inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to each element  $a$  satisfying  $p(a) = 0$ . Indeed, it is clear that  $p(a + P_k(x - a)) = p(a)$  for all  $a, x \in l^\infty(\mathbb{N})$ . So, if  $p(a) = 0$ , then we have that  $p(a + P_k(x - a)) = 0 \leq p(x)$  for all  $x \in l^\infty(\mathbb{N})$ . Thus,  $\inf_{k \in \mathbb{N}} p(a + P_k(x - a)) \leq p(x)$ , for all  $x \in l^\infty(\mathbb{N})$ . If  $p(a) \neq 0$ , then  $p(a + P_k(0 - a)) = p(a) > 0 = p(0)$  and so in this case  $p$  is not inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to  $a$ . □

Note that in the Banach space  $(l^\infty(\mathbb{N}), \|\cdot\|_\infty)$  the sequence  $e_n := (\delta_j^n)$  (where  $\delta_j^n$  is the Kronecker symbol), is not a topological basis since in general  $\|P_k(x) - x\|_\infty$  does not converge to 0, when  $k \rightarrow +\infty$ .

**Proposition 2.** *Let  $f : (l^\infty(\mathbb{N}), \|\cdot\|_\infty) \rightarrow \mathbb{R}$  be a  $L$ -Lipschitz continuous function ( $L \geq 0$ ) and  $p(x) = \limsup_k |x_k|$ . Then,  $f + Lp$  is inf-finitely determined with respect to each point  $a$  of  $c_0(\mathbb{N})$ .*

*Proof.* Since  $f$  is  $L$ -Lipschitz continuous, we have that for all  $x \in l^\infty(\mathbb{N})$

$$\begin{aligned} |f(a + P_k(x - a)) - f(x)| &\leq L \|a + P_k(x - a) - x\|_\infty \\ &= L \sup_{n \geq k+1} |a_n - x_n| \\ &\leq L \sup_{n \geq k+1} |a_n| + L \sup_{n \geq k+1} |x_n| \end{aligned}$$

Besides, we have that  $p(a + P_k(x - a)) = p(a) = 0$  since  $a \in c_0(\mathbb{N})$ . Thus, using the above inequality we get

$$f(a + P_k(x - a)) + Lp(a + P_k(x - a)) \leq f(x) + L \sup_{n \geq k+1} |a_n| + L \sup_{n \geq k+1} |x_n|.$$

Taking the limit superior over  $k \in \mathbb{N}$ , we get that

$$\limsup_{k \rightarrow +\infty} (f(a + P_k(x - a)) + Lp(a + P_k(x - a))) \leq f(x) + Lp(x).$$

and so,

$$\inf_{k \in \mathbb{N}} (f(a + P_k(x - a)) + Lp(a + P_k(x - a))) \leq f(x) + Lp(x).$$

Hence,  $f$  is inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to  $a \in c_0(\mathbb{N})$ .  $\square$

### 3. NECESSARY AND SUFFICIENT CONDITION OF CONVEX OPTIMALITY

Note that we can construct on  $l^\infty(\mathbb{N})$  canonical examples of convex inf-finitely determined functions at some point  $a \in l^\infty(\mathbb{N})$  (and norm continuous)

$$f : (l^\infty(\mathbb{N}), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$$

which are differentiable at  $a$  in the directions of the canonical basis  $(e_n)_{n \geq 1}$  of  $c_0(\mathbb{N})$  but are not Gâteaux-differentiable at this point. We proceed as follows: let  $g : (c_0(\mathbb{N}), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$  be a convex  $L$ -Lipschitz continuous function which is Gâteaux-differentiable but not Fréchet-differentiable at  $a \in c_0(\mathbb{N})$  (such function  $g$  always exists and can be constructed canonically, see for instance [2]). Let us define  $f : (l^\infty(\mathbb{N}), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$  by

$$f(x) := \inf_{y \in c_0(\mathbb{N})} \{g(y) + L(\|x - y\|_\infty + p(x - y))\},$$

where  $p(x) = \limsup_n |x_n|$  for all  $x \in l^\infty(\mathbb{N})$ . The function  $f$  is convex and Lipschitz continuous satisfying  $f|_{c_0(\mathbb{N})} = g$ , where  $f|_{c_0(\mathbb{N})}$  denotes the restriction of  $f$  to  $c_0(\mathbb{N})$ . It follows that  $f'(a; e_n) = g'(a; e_n)$  exists for all  $n \in \mathbb{N}$ . However,  $f$  cannot be Gâteaux-differentiable at  $a \in c_0(\mathbb{N})$ , otherwise  $f|_{c_0(\mathbb{N})} = g$  would be Fréchet-differentiable at  $a$  since the canonical embedding  $i : c_0(\mathbb{N}) \longrightarrow l^\infty(\mathbb{N})$  is a limited operator (see [2, Corollary 1] for details). Note also that  $f$  is inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to each point of  $a \in c_0(\mathbb{N})$ .

Thus, in infinite dimension, the fact that a convex continuous function  $f$  is differentiable at  $a$  in the directions  $(e_n)$  does not implies that  $f$  is Gâteaux-differentiable at  $a$  (see also Example 1).

**Definition 3.** (*Qualification condition*) Let  $E$  be a topological vector space equipped with a biorthogonal system  $(e_n, e_n^*)$  (not necessarily a topological basis). Let  $X \subset E$  be a non-empty subset of  $E$  and let  $a \in X$  be a fixed point of  $E$ . We say that the set  $X$  is qualified at  $a$  if the following conditions hold.

- For all  $n \in \mathbb{N}$ , there exists  $\alpha_n > 0$  such that  $a + te_n \in X$  for all  $|t| < \alpha_n$ .
- $P_k(X - a) \subset X - a$  for all  $k \in \mathbb{N}$ .

We define the space  $E_k$  as the image of  $E$  by  $P_k$ , that is,  $E_k = P_k(E)$ , which is a finite dimensional vector space isomorphic to  $\mathbb{R}^k$ . Let  $X$  be a subset of  $E$ . For all  $k \in \mathbb{N}$ , we denote  $X_k := P_k(X)$  and by  $Int_{E_k}(X_k)$  we mean the relative interior of  $X_k$ , that is the interior of  $X_k$  in  $E_k \simeq \mathbb{R}^k$ .

*Remark 3.* Provided that  $X$  is a convex set, the qualification condition implies that  $P_k(a) \in \text{Int}_{E_k}(X_k)$  for all  $k \in \mathbb{N}^*$ , but is in general weaker than the fact that  $a \in \text{Int}_E(X)$ . Indeed, let  $E := (l^1(\mathbb{N}), \|\cdot\|_1)$  and let  $X_+ := \{(x_n) \in l^1(\mathbb{N}) : x_n > 0; \forall n \in \mathbb{N}\}$  be the convex positive cone of  $l^1(\mathbb{N})$ . Then,

- $\text{Int}(X_+) = \emptyset$ ,
- however,  $X_+$  is qualified at each of its points.

We give below the main result of this section which gives a necessary and sufficient condition of optimality by using the notion of inf-finitely determined function. The proof is based on a reduction to the finite dimension. For recent works on convex optimization in finite dimension, we refer for instance to [6] and [15].

**Theorem 3.** *Let  $E$  be a topological vector space equipped with a biorthogonal system  $(e_n, e_n^*)$  (not necessarily a topological basis). Let  $X \subset E$  be a non-empty convex subset of  $E$  and let  $a \in X$ . Suppose that  $X$  is qualified at  $a$ . Let  $f : X \rightarrow \mathbb{R}$  be a convex function, such that  $f$  is inf-finitely determined on  $X$  with respect to  $a$  and differentiable at  $a$  in the directions  $(e_n)$ . Then, the following assertions are equivalent.*

- (a)  $f(a) = \inf_{x \in X} f(x)$
- (b)  $f'(a, e_n) = 0, \quad \forall n \in \mathbb{N}$

*Proof.* The part (a)  $\implies$  (b) is easy. Indeed, suppose that  $f(a) = \inf_{x \in X} f(x)$ . Then, we have that

$$0 \leq f(x) - f(a) \quad \forall x \in X.$$

In particular, since  $X$  is qualified at  $a$ , for all  $n \in \mathbb{N}$  there exists  $\alpha_n > 0$  such that for all  $|t| < \alpha_n$ , we have that  $a + te_n \in X$  and so

$$0 \leq f(a + te_n) - f(a).$$

Thus, we get that  $0 \leq \lim_{t \rightarrow 0^+} \frac{f(a + te_n) - f(a)}{t} = f'(a; e_n)$ . Similarly, we have  $0 \geq \lim_{t \rightarrow 0^-} \frac{f(a + te_n) - f(a)}{t} = f'(a; e_n)$ . Hence,  $f'(a, e_n) = 0, \quad \forall n \in \mathbb{N}$ .

Now, we prove (b)  $\implies$  (a). Let us define  $f_k : X_k \subset E_k \rightarrow \mathbb{R}$  as follows: for all  $x \in X$ ,

$$f_k(P_k(x)) := f(a + P_k(x - a)).$$

Note that  $f_k$  is well defined and that  $P_k(a) \in \text{Int}_{E_k}(X_k)$  for all  $k \in \mathbb{N}$  (by the qualification condition of  $X$  at  $a$ , see Remark 3). We prove that, for all  $k \in \mathbb{N}$ , the convex function  $f_k$  is Fréchet-differentiable at  $P_k(a)$ . Indeed, for all  $n \leq k$  we have that  $P_k(e_n) = e_n$  and we have that  $f_k(P_k(a)) = f(a)$ . Thus, for all  $n \leq k$  and all small  $t$  we have

$$\begin{aligned} (4) \quad f_k(P_k(a) + te_n) - f_k(P_k(a)) &= f_k(P_k(a + te_n)) - f_k(P_k(a)) \\ &= f(a + te_n) - f(a). \end{aligned}$$

It follows that

$$(5) \quad f'_k(P_k(a); e_n) = f'(a; e_n).$$

This shows that  $f'_k(P_k(a); e_n)$  exists for each  $e_n \in E_k$ ,  $n \in \{1, \dots, k\}$ . Since  $f_k$  is a convex function on the convex set  $X_k$ ,  $P_k(a) \in \text{Int}_{E_k}(X_k)$  and since  $E_k$  is of finite dimension with  $(e_n)_{1 \leq n \leq k}$  as a basis, then it is well known (see [11, Theorem 6.1.1]) that  $f_k$  is Fréchet-differentiable at  $P_k(a)$ .

Thanks to the equations (b) and (5), we have that for all  $k \in \mathbb{N}$ ,

$$Df_k(P_k(a)) = 0,$$

where  $Df_k(P_k(a))$  denotes the Fréchet-derivative of  $f_k$  at  $P_k(a)$ . Moreover,  $f_k$  is a convex function defined on the convex set  $X_k \subset E_k$  and  $P_k(a) \in \text{Int}_{E_k}(X_k)$  (by the qualification condition). It follows that

$$(6) \quad f_k(P_k(a)) = \inf_{y \in X_k} f_k(y).$$

For all  $x \in X$  and all  $k \in \mathbb{N}$ , we have that  $P_k(x) \in X_k$ , then by using (6) we get

$$f(a) = f_k(P_k(a)) = \inf_{y \in X_k} f_k(y) \leq f_k(P_k(x)) := f(a + P_k(x - a)).$$

Since  $f$  is inf-finitely determined on  $X$  with respect to the point  $a$ , then by taking the infimum in the above inequality we obtain that for all  $x \in X$

$$f(a) \leq \inf_{k \in \mathbb{N}} f(a + P_k(x - a)) \leq f(x).$$

It follows that  $f(a) = \inf_{x \in X} f(x)$ . □

*Remark 4.* The example of the (norm) continuous seminorm  $p : l^\infty(\mathbb{N}) \rightarrow \mathbb{R}$ ,  $x \mapsto \limsup_n |x_n|$  shows that the condition of inf-finitely determined property cannot be dropped from the hypothesis of Theorem 3. Indeed, we know that for each  $a \in l^\infty(\mathbb{N})$ , we have that  $p'(a, e_n) = 0$  for all  $n \in \mathbb{N}^*$  (see Example 1). On the other hand, if  $p(a) \neq 0$ , then clearly  $a$  is not a minimum for  $p$ . Thus, Theorem 3 does not apply for  $p$  at  $a$  if  $p(a) \neq 0$ . This is due to the fact that  $p$  is not inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to  $a$  if  $p(a) \neq 0$ . However, if  $p(a) = 0$ , then  $p$  is inf-finitely determined on  $l^\infty(\mathbb{N})$  with respect to  $a$ . In this case Theorem 3 applies and  $p$  has a minimum at  $a$  (which is trivial here since  $p(a) = 0 \leq p(x)$  for all  $a \in c_0(\mathbb{N})$  and all  $x \in l^\infty(\mathbb{N})$ ).

The above Theorem shows that, for a convex function which is inf-finitely determined with respect to  $a \in E$  and differentiable at  $a$  in the directions  $(e_n)_{n \geq 0}$ , a necessary and sufficient condition to have a minimum at  $a$  is to satisfy  $f'(a, e_n) = 0$ ,  $\forall n \in \mathbb{N}$ . In several examples, it is easy to calculate the derivative  $f'(a, e_n)$  and also to solve  $f'(a, e_n) = 0$ ,  $\forall n \in \mathbb{N}$ . Thus, the candidate for the minimum can be exhibited. Since the condition is also sufficient, we get the points that realizes the minimum (see Section 5 for examples). Moreover, in infinite dimension, the differentiability of  $f$  in the directions  $(e_n)$  at some point  $a$ , does not implies in general its Gâteaux-differentiable at  $a$ . An example in the space  $l^\infty(\mathbb{N})$  illustrating this situation

was given in Example 1. Thus, Theorem 3 can be applied for instance in  $E = l^\infty(\mathbb{N})$  without the Gâteaux-differentiability assumption. For example, combining Proposition 2 and Theorem 3, we get the following corollary:

**Corollary 3.** *Let  $f : (l^\infty(\mathbb{N}), \|\cdot\|_\infty) \rightarrow \mathbb{R}$  be a convex  $L$ -Lipschitz continuous function ( $L \geq 0$ ) and  $p(x) = \limsup_k |x_k|$ . Suppose that there exists  $a \in c_0(\mathbb{N})$  such that  $f'(a, e_n) = 0$  for all  $n \in \mathbb{N}$ . Then,  $f + Lp$  has a minimum on  $l^\infty(\mathbb{N})$  at  $a$ .*

However, in Hausdorff locally convex topological vector spaces equipped with a biorthogonal system  $(e_n, e_n^*)$ , where  $(e_n)$  is a topological basis, the situation is different. Indeed, as we show it in Corollary 4, in this situation, the differentiability of a convex continuous function  $f$  in the directions  $(e_n)$  at some point  $a$ , is equivalent to the Gâteaux-differentiability of  $f$  at  $a$ . This result is a natural extension of a well-known result concerning the Gâteaux-differentiability of convex functions in finite dimension (see [11, Theorem 6.1.1]). Note that this result applies even if  $E$  is not a normed space like the Fréchet space  $(\mathbb{R}^\mathbb{N}, d_{\mathbb{R}^\mathbb{N}})$  of all real sequences, equipped with the distance: for all  $x = (x_n)$  and  $y = (y_n)$ ,

$$d_{\mathbb{R}^\mathbb{N}}(x, y) := \sum_{i=1}^{+\infty} \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|}$$

**Corollary 4.** *Let  $E$  be a Hausdorff locally convex topological vector space equipped with a biorthogonal system  $(e_n, e_n^*)$ , where  $(e_n)$  is a topological basis. Let  $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function. Suppose that  $f$  is finite and upper semicontinuous at  $a \in E$  and that  $f'(a; e_n)$  exists for all  $n \in \mathbb{N}$  with  $\partial f(a) \neq \emptyset$ . Then,  $\partial f(a)$  is a singleton. In consequence, if  $f$  is convex and continuous at  $a$ , then  $f'(a; e_n)$  exists for all  $n \in \mathbb{N}$ , if and only if  $f$  is Gâteaux-differentiable at  $a$ .*

*Proof.* Suppose that  $f'(a; e_n)$  exists for all  $n \in \mathbb{N}$  and let  $p, q \in \partial f(a)$ . Then,  $a$  is a minimum of the functions  $f - p$  and  $f - q$ . On the other hand, clearly  $E$  is qualified at  $a$  and the functions  $f - p$  and  $f - q$  are inf-finitely determined on  $E$  with respect to  $a$  since they are upper semicontinuous at this point. Thus, applying Theorem 3, once to  $f - p$  and again to  $f - q$ , we obtain that  $\langle p, e_n \rangle = f'(a, e_n) = \langle q, e_n \rangle$ ,  $\forall n \in \mathbb{N}$ . It follows that  $p = q$  since  $p, q \in E^*$  and  $(e_n)_{n \geq 1}$  is a topological basis. Thus,  $\partial f(a)$  is a singleton. If in addition  $f$  is convex continuous at  $a$  then we know from [12, Proposition 10.c, p.60,] that  $\partial f(a) \neq \emptyset$ . It follows that  $\partial f(a)$  is a singleton. To conclude, we know from [12, Corollary 10.g, p. 66] that  $f$  is Gâteaux-differentiable at  $a$  if and only if  $\partial f(a)$  is a singleton.  $\square$

It is well known (see for instance [14, Examples 1.4]) that the norm of  $l^1(\mathbb{N})$ ,  $\|x\|_1 = \sum_{n \geq 0} |x_n|$  is Gâteaux-differentiable at  $x = (x_n)$  if and only if  $x_n \neq 0$

for all  $n \in \mathbb{N}$ . This fact is a particular case of a more general result given in the following proposition, which is a consequence of Corollary 4. Indeed, it suffices to take  $u_n(t) = |t|$  for all  $t \in \mathbb{R}$  and all  $n \in \mathbb{N}$  in the following proposition, to see more simply why, the norm  $\|\cdot\|_1$  is Gâteaux-differentiable at  $x = (x_n)$  if and only if  $x_n \neq 0$  for all  $n \in \mathbb{N}$ .

**Proposition 3.** *Let  $(E, \|\cdot\|)$  be a Banach space having a Schauder basis  $(e_n)$  and let  $(e_n, e_n^*)$  be a biorthogonal system. For each  $n \in \mathbb{N}$ , let  $u_n : \mathbb{R} \rightarrow \mathbb{R}$  be a convex continuous function. Suppose that the series  $\sum_{n=1}^{+\infty} u_n(\langle e_n^*, \cdot \rangle)$  converges pointwise to a real valued continuous function  $f$ . Then,*

(i)  *$f$  is Gâteaux-differentiable at  $x \in E$ , if and only if, for all  $n \in \mathbb{N}$  the function  $u_n$  is differentiable at  $\langle e_n^*, x \rangle$ . In this case, we have that for all  $h \in E$ ,*

$$Df(x)(h) = \sum_{n=1}^{+\infty} \langle e_n^*, h \rangle u_n'(\langle e_n^*, x \rangle),$$

where  $Df(x)$  denotes the Gâteaux-derivative of  $f$  at  $x$ .

(ii) *the set of points at which  $f$  is not Gâteaux-differentiable is a countable union of affine hyperplanes.*

*Proof.* (i) It is clear that for each  $n \in \mathbb{N}$ , we have that  $f'(x, e_n)$  exists if and only if  $u_n$  is differentiable at  $\langle e_n^*, x \rangle$ , in this case  $f'(x, e_n) = u_n'(\langle e_n^*, x \rangle)$ . Thus, we conclude using Corollary 4.

(ii) It is well known that a convex continuous function from  $\mathbb{R}$  to  $\mathbb{R}$  is differentiable at all but (at most) countably many points of  $\mathbb{R}$  (see [14, Theorem 1.16.]). Thus, for each  $n \in \mathbb{N}$ , the set

$$C_n := \{t \in \mathbb{R} : u_n'(t) \text{ does not exist}\},$$

is at most a countable subset of  $\mathbb{R}$ . Using part (i), we clearly see that  $f$  is not Gâteaux-differentiable at  $x \in E$  if and only if  $x \in \cup_{n \in \mathbb{N}} \cup_{t \in C_n} (e_n^*)^{-1}(\{t\})$ . Finally, it is clear that  $(e_n^*)^{-1}(\{t\})$  is an affine hyperplane for each  $n \in \mathbb{N}$  and each  $t \in C_n$ .  $\square$

#### 4. APPLICATION TO THE KARUSH-KUHN-TUCKER THEOREM

We follow the notation given in [3]. Let  $E$  be a real linear space and  $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$  be a given function. Consider the minimizing problem for the function  $f$  on a subset  $A_E \subset E$ , that is, the problem

$$(\mathcal{P}) \quad \min\{f(x) : x \in A_E\}.$$

The set  $A_E$  constitutes the constraints of Problem  $(\mathcal{P})$ . We say that an element  $\bar{x} \in E$  is feasible if  $\bar{x} \in A_E \cap \text{dom}(f)$ . The mathematical programming problem  $(\mathcal{P})$  is said to be consistent if  $A_E \cap \text{dom}(f) \neq \emptyset$ , that is, if it has feasible elements. A feasible element  $x_0$  is called an optimal solution of  $(\mathcal{P})$  if

$$f(x_0) = \inf\{f(x) : x \in A_E\}.$$



The subset  $A_E$  is often defined by the solutions of a finite number of inequalities as in

$$A_E := \{x \in E : g_i(x) \leq 0, \forall i = 1, \dots, m\},$$

where  $g_i$  are extended real-valued functions on  $E$ . Let us set

$$E_0 := \text{dom}(f) \cap_{i=1}^m \text{dom}(g_i).$$

We call Slater's constraint qualification, the following condition:

(S) There exists a point  $\bar{x} \in A_E$  such that  $g_i(\bar{x}) < 0, \forall i = 1, 2, \dots, m$ .

In the following corollary, we give a Karush-Kuhn-Tucker theorem in infinite dimension, where Gâteaux-differentiability is replaced by the weaker condition of differentiability in the directions of  $(e_n)$ .

**Corollary 5.** *Let  $E$  be a topological vector space equipped with a biorthogonal system  $(e_n, e_n^*)$  (not necessarily a topological basis). Let  $f, g_1, \dots, g_m : E \rightarrow \mathbb{R} \cup \{+\infty\}$  be convex functions. Suppose that  $E_0$  is qualified at  $x_0 \in A_E$  and that  $f, g_1, \dots, g_m$  are finitely determined functions on  $E_0$  with respect to  $x_0$  and differentiable at  $x_0$  in the directions  $(e_n)$ . Then, we have (1)  $\implies$  (2). If moreover, the Slater's condition (S) is satisfied, then (1)  $\iff$  (2).*

(1) There exists  $\lambda_i^* \geq 0$  for all  $i \in \{1, \dots, m\}$  such that

$$\lambda_i^* g_i(x_0) = 0, \forall i \in \{1, 2, \dots, m\}$$

$$f'(x_0, e_n) + \sum_{i=1}^m \lambda_i^* g_i'(x_0, e_n) = 0, \quad \forall n \in \mathbb{N}^*$$

(2)  $f(x_0) = \inf\{f(x), x \in A_E\}$ .

*Proof.* (1)  $\implies$  (2). We apply Theorem 3 to the function  $\tilde{f} = f + \sum_{i=1}^m \lambda_i^* g_i$  which is finitely determined on  $E_0$  with respect to  $x_0$  and differentiable at  $x_0$  in the directions  $(e_n)$  with  $\tilde{f}'(x_0; e_n) = 0$  for all  $n \in \mathbb{N}^*$ , to get that for all  $x \in E_0$

$$f(x_0) + \sum_{i=1}^m \lambda_i^* g_i(x_0) \leq f(x) + \sum_{i=1}^m \lambda_i^* g_i(x).$$

Since,  $\lambda_i^* g_i(x_0) = 0$  for all  $i \in \{1, 2, \dots, m\}$  by hypothesis, then for all  $x \in A_E \cap E_0$ , we obtain that

$$\begin{aligned} f(x_0) &\leq f(x) + \sum_{i=1}^m \lambda_i^* g_i(x) \\ &\leq f(x) \quad (\text{Since } \forall x \in A_E : \lambda_i^* \geq 0; g_i(x) \leq 0). \end{aligned}$$

Hence,  $f(x_0) = \inf\{f(x), x \in A_E \cap E_0\} = \inf\{f(x), x \in A_E\}$ .

(2)  $\implies$  (1). If moreover (S) is satisfied, then the implication (2)  $\implies$  (1) follows easily from [3, Theorem 3.4].  $\square$

## 5. EXAMPLES

As proved in Example 1, in infinite dimension, the fact that a convex continuous function  $f$  is differentiable at  $a$  in the directions  $(e_n)_{n \geq 1}$  does not imply that  $f$  is Gâteaux-differentiable at  $a$ . We give simple examples showing how Theorem 3 can be applied by using only differentiability in the directions  $(e_n)_{n \geq 1}$ .

*Example 2.* Let  $f : (l^\infty(\mathbb{N}), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$  be the convex continuous function defined by

$$f(x) = \limsup |x_n| + \sum_{n=1}^{+\infty} \beta^n (x_n^2 - \frac{x_n}{n}),$$

where,  $0 < \beta < 1$  is a fixed real number. We prove that the problem  $\inf_{x \in l^\infty(\mathbb{N})} f(x)$  has a unique solution, that is  $a = (\frac{1}{2n})_{n \geq 1}$ .

*Proof.* The qualification condition is trivial at each point. On the other hand, we have that  $f'(x; e_n) = p'(x; e_n) + \beta^n (2x_n - \frac{1}{n}) = \beta^n (2x_n - \frac{1}{n})$  for all  $n \in \mathbb{N}^*$  (where  $p(x) = \limsup |x_n|$ , see Example 1). It follows that  $f'(x; e_n) = 0$  if and only if  $x = (\frac{1}{2n})$ . In this case  $p((\frac{1}{2n})) = 0$  and so  $p$  is inf-finitely determined with respect to this point (see Example 1). On the other hand,  $x \mapsto f_1(x) = \sum_{n=1}^{+\infty} \beta^n (x_n^2 - \frac{x_n}{n})$  is finitely determined on  $l^\infty(\mathbb{N})$  with respect to  $(\frac{1}{2n})$ . Indeed, let  $a, x \in l^\infty(\mathbb{N})$ , we have

$$\begin{aligned} |f_1(a + P^k(x - a)) - f_1(x)| &= \left| \sum_{n=k+1}^{+\infty} \beta^n [(a_n^2 - \frac{a_n}{n}) - (x_n^2 - \frac{x_n}{n})] \right| \\ &\leq C \sum_{n=k+1}^{+\infty} \beta^n, \end{aligned}$$

where  $C$  is a positive real number depending only on  $a$  and  $x$ . Thus,  $f_1$  is finitely determined on  $l^\infty(\mathbb{N})$  with respect to each  $a$ , and so  $f$  is inf-finitely determined with respect to  $(\frac{1}{2n})$ . Then, we can apply Theorem 3. Hence the sequence  $a = (\frac{1}{2n})$  is the unique optimal solution of the problem  $\inf_{x \in l^\infty(\mathbb{N})} f(x)$ . Note that  $f$  is not Gâteaux-differentiable at  $(\frac{1}{2n})$  since  $p((x_n)) = \limsup |x_n|$  is nowhere Gâteaux-differentiable (see Example 1).  $\square$

*Example 3.* Let  $E = \mathbb{R}^{\mathbb{N}}$  and  $X := l^1(\mathbb{N}) \cap (\mathbb{R}_+)^{\mathbb{N}}$  (convex subset) and let  $f : X \longrightarrow \mathbb{R}$  be the convex function defined by

$$f((x_n)_n) = \sum_{n=0}^{+\infty} x_n - \sum_{n=0}^{+\infty} 2\beta^n x_n^{\frac{1}{2}}$$

(where  $0 < \beta < 1$  is a fixed real number). The problem is to minimize  $f$  on  $X$ . A solution of this problem is  $a = (\beta^{2^n}) \in X$ .

*Proof.* The function  $f$  is differentiable in the directions  $(e_n)_{n \geq 1}$  at each  $x = (x_n) \in X$  such that  $x_n > 0$  for all  $n \in \mathbb{N}$  and we have  $f'(x; e_n) = 1 - \frac{\beta^n}{(x_n)^{\frac{1}{2}}}$  for all  $n \in \mathbb{N}$ . Now, suppose that  $f'(x; e_n) = 0$  for all  $n \in \mathbb{N}$ . Then, we have  $x_n = \beta^{2^n}$  for all  $n \in \mathbb{N}$ . Clearly, the point  $a = (\beta^{2^n})$  belongs to  $X$ . To show that  $a$  is an optimal solution of the problem of minimization, it suffices to prove that  $X$  is qualified at  $(\beta^{2^n})$  and that  $f$  is inf-finitely determined on  $X$  with respect to  $(\beta^{2^n})$ . In fact,  $X$  is qualified at each point  $(x_n)$  such that  $x_n > 0$  for all  $n \in \mathbb{N}$  (easy to see) and  $f$  is finitely determined on  $X$  with respect to each point  $x$  of  $X$ . Indeed, let  $x, a \in X$ , then

$$f(a + P^k(x - a)) - f(x) = \sum_{n=k+1}^{\infty} (a_n - x_n) - \sum_{n=k+1}^{\infty} 2\beta^n((a_n)^{\frac{1}{2}} - x_n^{\frac{1}{2}}).$$

It follows that  $\lim_{k \rightarrow +\infty} f(a + P^k(x - a)) = f(x)$  since  $a - x \in l^1(\mathbb{N})$ . Hence,  $f$  is finitely determined on  $X$  with respect to each point  $x$  of  $X$  in particular with respect to the point  $a = (\beta^{2^n})$ .  $\square$

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