Crisis Risk Prediction with Concavity from Polymodel

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Abstract

Financial crises is an important research topic because of their impact on the economy, the businesses and the populations. However, prior research tend to show systemic risk measures which are reactive, in the sense that risk surges after the crisis starts. Few of them succeed in predicting financial crises in advance. In this paper, we first introduce a toy model based on a dynamic regime switching process producing normal mixture distributions. We observe that the relative concavity of various indices increases before a crisis. We use this stylized fact to introduce a measure of concavity from nonlinear Polymodel, as a crisis risk indicator, and test it against known crises. We validate the indicator by using it for a trading strategy that holds long or short positions on S&P 500, depending on the indicator value.

***Keywords*—** crisis risk, financial crisis, concavity, Polymodel, trading strategy

1. INTRODUCTION

Financial crises is an important research topic because of their impact on the economy, the businesses and the populations. One of the most famous financial crises was 2007-2008 financial crisis. Some most famous financial institutions collapsed, nationalized, or survived only with massive government support. This crisis impacted financial institutions across the whole world. [1] It influenced not only financial area, but also generated a collapse of trades in almost all areas, and lead to evictions, foreclosures, and prolonged unemployment. Companies in different areas bankrupted and a lot of people lost their wealth and jobs in different countries. Because of that crisis, states and financial institutions made policies and regulations to prevent another possible collapse of the world financial system. However, financial crises may still be caused by many reasons.

With the process of globalization, the global markets are becoming more and more correlated. The potential financial crises in the future may influence more countries and people. The aim of our research is to find indicators of crisis risk, so financial institutions and investors can reduce their losses, and even make profits during financial crises.

In the past years, financial analysts tried different crisis risk measures. In paper [2], the authors calculated correlation between every pair of stocks. Then they calculated the mean and median of the correlations every month, and used them to measure crisis risk. Correlation between returns did increase during crises. However, correlation seems to increase after crises start, not before crises.

The CBOE Volatility Index, whose ticker is VIX, is a measure of the stock market's expectation of volatility implied by S&P 500 index options. S&P 500 is an index composed of about 500 large companies in the United States, so its performance can represent the U.S. stock market. People can use its volatility index, VIX to measure the volatility of the market. The higher VIX, the more volatile and riskier of the market. During crises, VIX is higher than normal periods. However, some critics said that, the predictive power of most volatility forecasting models is similar to simple past volatility. [3] [4]

Brownlees and Engle created a risk measure SRISK: the company's capital shortfall conditional on a crisis. It is a function of a company's size, leverage and risk. The total SRISK in the market can measure the systemic risk. However, SRISK works well on highly leveraged financial institutions. It may not work well on companies in other industries. [5]

Sornette and Andersen applied a nonlinear super-exponential rational model in the market. They said the exponent can detect the speculative bubbles and herding behavior in the market. [6]

Another popular model to predict market risk is Regime Switching Model. Financial markets often change their behavior patterns. The changes may happen abruptly, and every pattern often persists for a long time period. At the start of a crisis, the mean, volatility, and correlation patterns in stock market usually changes dramatically. After the crisis ends, the pattern changes again. Since one pattern often lasts for a long period, regime switching models can learn the characteristics of different patterns, and capture sudden changes of behavior.

Regime Switching model assumes that there are k different market states (regimes). Market states are generally modeled through a discrete variable $s_t \in \{0,1,\ldots,k\}$. s_t tracks the particular regime at time t. Returns' distributions are decided by the value of s_t.

Regime switching models are popular for several reasons. First, the idea of regime changes is natural and intuitive. When applied to financial series, different regimes often correspond to different periods in policies or financial behavior.

Second, large amounts of non-linear effects can be generated in regime switching model. Regime switching model is actually a mixture of several distributions. It draws the distributions in several market states, and assigns probabilities that a financial series is in each market state. By appropriately mixing normal or other common types of distributions, large amounts of non-linear effects can be generated. So regime switching model can capture stylized behavior of many financial return series, including fat tails, volatility clusters, skewness, and time-varying correlations. Even though it is hard to know the true distribution in the market, regime switching models can approximate its behavior in the market with basic structure distributions. What is more, based on the basic distributions, people can derive formulas or properties in the complex market analytically. [7]

Hidden Markov Model (HMM) is the most common method people use to solve regime switching problem. They assume that s_t follows a homogenous first-order Markov Chain and $\Pi_{[i,j]} = P(s_t = j | s_{t-1} = i) = p_{ij}$ is a constant. It is called "hidden" because the market states are not observable.

One disadvantage of HMM is that the relation among state series in the reality is not Markov, and then the transition matrix may not be fixed. Another important issue in regime switching model, including HMM, is specifying the number of regimes. It is hard to implement tests on the number of regimes in practice, so the number fixed in the model may be wrong.

Considering those problems in traditional regime switching model, we want to build a model that can also capture nonlinear effects in the market, but do not have those limits. In this paper, we use concavity statistics got from Polymodel to achieve those goals. Polymodel is a technique to measure the amount of nonlinearity among a set of indices by performing nonlinear regressions of indices with respect to one another.

We assume that returns distribution is a normal mixture, but the probability of the market falling in different regimes is more varied and complex than HMM. Instead of figuring out the transition matrix, we measure the concavity change in the market and use that to indicate when the market approaches a crisis. With our model, we do not need to assume the regimes reoccur or transition matrix is fixed like HMM. We can capture the start of a crisis, no matter how many and what states the market has, what is the current market state. Those are the advantages of our model over Hidden Markov Model.

In Section 2.1, we derive how the concavity in the market changes before a crisis. In Section Polymodel, we introduce Polymodel and how we measure concavity in the market from it. In Section Empirical Crisis Indicator, we implement experiments with indices data, and show how the market concavity changes before a crisis in practice. We build a crisis indicator based on the concavity statistics. In Section Trading Strategies, we construct trading strategies with the crisis indicator. In Section Conclusions, we summarize this paper.

2. THEORETICAL DERIVATION

2.1 Concavity v.s. Crisis Risk

We are going to find the relationship between the concavity and crisis risk in this section. Let us build a toy model first. We assume that there are two regimes R_1 and R_2. R_1 is the state of normal market, and R_2 corresponds to the market during a crisis. X and Y are two returns in the market.

A stylized fact of returns is that correlations increase during financial crises, as shown by Longin and Solnik [8], Ang and Chen [9], and other researchers. With regard to that fact, we make several assumptions in our toy model.

1. R_1 is the market state of normal market. Under R_1, $X \sim \mathcal{N}(0,\, \sigma_X^2)$, $Y \sim \mathcal{N}(0,\, \sigma_Y^2)$. $Corr(X, Y) = \rho$.

2. R_2 is the market state of a crisis. Under R_2, $X \sim \mathcal{N}(0,\, \sigma_X'^2)$, $Y \sim \mathcal{N}(0,\, \sigma_Y'^2)$. $Corr(X, Y) = \rho'$.

3. Since variances of returns and correlations between returns usually increase during crises, we assume that $\sigma_X' > \sigma_X$, $\sigma_Y' > \sigma_Y$, $\rho' > \rho$. And $\frac{\sigma_Y'}{\sigma_X'}\rho' > \frac{\sigma_Y}{\sigma_X}\rho$.

4. The true probabilities of the market being in each state are $P(R_1) = p_1$, and $P(R_2) = p_2$. p_1 and p_2 change over time. If the market is not in a crisis, $p_1 >> p_2$.

According to the regression formula and the first two assumptions, under R_1, $Y = \beta X + \epsilon$, where $\beta = \frac{\sigma_Y}{\sigma_X}\rho$, and $E[\epsilon] = 0$. $E[Y] = \beta E[X]$. Similarly, under R_2, $E[Y] = \beta' E[X]$, where $\beta' = \frac{\sigma_Y'}{\sigma_X'}\rho'$. From the third assumption, $\beta' > \beta$.

Under our toy model, we have $E[Y|X = x] = E[Y|X = x]_{R_1} P(R_1|X = x) + E[Y|X = x]_{R_2} P(R_2|X = x)$. $P(R_i|X = x)$ means the probability that the market is in regime R_i given that $X = x$. It changes over time even if x is fixed.

$$
\begin{aligned}
P(R_1|X=x) &= \frac{P(R_1, X=x)}{P(X=x)} \\
&= \frac{P(X=x|R_1)P(R_1)}{P(X=x|R_1)P(R_1)+P(X=x|R_2)P(R_2)} \\
&= \frac{1/(\sqrt{2\pi}\sigma_X)\exp(-x^2/(2\sigma_X^2))p_1}{1/(\sqrt{2\pi}\sigma_X)\exp(-x^2/(2\sigma_X^2))p_1 + 1/(\sqrt{2\pi}\sigma'_X)\exp(-x^2/(2\sigma'^2_X))p_2} \\
&= \frac{1}{1+\frac{\sigma_X p_2}{\sigma'_X p_1}\exp\left(-\frac{x^2}{2\sigma'^2_X}+\frac{x^2}{2\sigma_X^2}\right)}
\end{aligned}
\tag{1}
$$

Similarly, we can get equation for $P(R_2|X=x)$, and then the equation for $E[Y|X=x]$. Under our model, we want to convert $E[Y|X=x]$ to some polynomial equation, and analyze how concavity changes when the crisis risk increases. We use the integral of the negative part of the second derivative of the polynomial between Y and X to measure the concavity. So after converting $E[Y|X=x]$ to some polynomial equation (to simplify, we use the third-degree polynomial to approximate), we are going to derive its second derivative function, and figure out how it changes when p_2 increases.

Firstly, let us convert $E[Y|X=x]$ into a third-degree polynomial equation. Considering $E[Y|X=x]_{R_1} = \beta x$ is a first- degree polynomial of x, we just need to convert $P(R_1|X=x)$ and $P(R_2|X=x)$ to second- degree polynomials of x.

Suppose $f(z) = P(R_1|X=x) = \frac{1}{1+ae^z}$, where $a = \frac{\sigma_X p_2}{\sigma'_X p_1}$, $z = -\frac{x^2}{2\sigma'^2_X} + \frac{x^2}{2\sigma_X^2} = \frac{\sigma'^2_X - \sigma_X^2}{2\sigma_X^2\sigma'^2_X}x^2 = \frac{\sigma'^2_X - \sigma_X^2}{\sigma'^2_X}\frac{x^2}{2\sigma_X^2}$. We want to convert $P(R_1|X=x)$ to a second-degree polynomial of x, which is equivalent to converting $f(z)$ to a first-degree polynomial of z. Therefore we only need linear approximation of $f(z)$.

2.2 Chebyshev Approximation

There are many methods of polynomial approximation. We use Chebyshev approximation, because the Chebyshev approximation formula is very close to the minimax polynomial, which has the smallest maximum deviation from the true function.

Chebyshev approximation of a function g using the first kind of Chebyshev polynomials is:

$$
P_n(s) = \frac{1}{2}c_0 + \sum_{k=1}^{n} c_k\, T_k(s) \tag{2}
$$

$$c_k = \frac{2}{n+1}\sum_{j=0}^{n} g\,(s_j)T_k(s_j), s_j = cos((j+\frac{1}{2})\pi/(n+1))$$

Using that approximation, the error is spread smoothly over $[-1,1]$. [10]

In our case, $z = \frac{\sigma_X'^2 - \sigma_X^2}{\sigma_X'^2}\frac{x^2}{2\sigma_X^2}$. Since $\sigma_X'^2 > \sigma_X^2$, the first part in z is smaller than 1. The second part is $2[x/(2\sigma_X)]^2$. σ_X is the standard deviation of X under R_1, and X follows normal distribution $\mathcal{N}(0, \sigma_X^2)$. So 95% of x values should be generated within $\pm 2\sigma_X$. And then at least 95% of z values should fall into interval [0,2]. Let us suppose $s = z - 1$, then $f(z) = \frac{1}{1+ae^z} = \frac{1}{1+a'e^s} = g(s)$, $a' = ae$, and 95% of s values are generated within $[-1,1]$.

We can apply Chebyshev approximation on $g(s) = \frac{1}{1+a'e^s}$. We only need the linear approximation,

$$P_1(s) = \frac{1}{2}c_0 + c_1 s \tag{3}$$

$$c_0 = \frac{1}{1+a'e^{\sqrt{2}/2}} + \frac{1}{1+a'e^{-\sqrt{2}/2}}$$

$$c_1 = \frac{\sqrt{2}}{2}(\frac{1}{1+a'e^{\sqrt{2}/2}} - \frac{1}{1+a'e^{-\sqrt{2}/2}})$$

And

$$E[Y|X=x]_{R_1}P(R_1|X=x) \approx (\frac{c_0}{2} + c_1 s)\beta x = (\frac{c_0}{2} + c_1(dx^2-1))\beta x \tag{4}$$

$d = \frac{\sigma_X'^2 - \sigma_X^2}{2\sigma_X^2\sigma_X'^2}$ is a constant.

Suppose $h(x) = E[Y|X=x]_{R_1}P(R_1|X=x) \approx (\frac{c_0}{2} + c_1(dx^2-1))\beta x = (\frac{c_0}{2} - c_1)\beta x + c_1 d\beta x^3$. Since we want to use the second derivative smaller than 0 part to measure concavity of $h(x)$, we are going to calculate the second derivative.

$$
\begin{aligned}
h''(x) &= 6c_1 d\beta x \\
&= 3\sqrt{2}\left(\frac{1}{1+a'e^{\sqrt{2}/2}} - \frac{1}{1+a'e^{-\sqrt{2}/2}}\right)d\beta x \\
&= 3\sqrt{2}\left(\frac{1}{1+a'e^{\sqrt{2}/2}} - \frac{e^{\sqrt{2}/2}}{e^{\sqrt{2}/2}+a'}\right)d\beta x \\
&= 3\sqrt{2}\frac{a' - a'e^{\sqrt{2}}}{(1+a'e^{\sqrt{2}/2})(a'+e^{\sqrt{2}/2})}d\beta x
\end{aligned}
\tag{5}
$$

Similarly, suppose $l(x) = E[Y|X=x]_{R_2}P(R_2|X=x)$, we can get $l(x) = \frac{1}{1+1/ae^{-z}}\beta' x$. Most values of $-z$ belongs to $[-2,0]$. Suppose $s' = -1-z$, we have $l(x) = \frac{1}{1+1/a\prime e^{s\prime}}\beta' x = g'(s')$. Applying Chebyshev approximation on $g'(s')$, we have:

$$
c'_0 = \frac{1}{1+1/a'e^{\sqrt{2}/2}} + \frac{1}{1+1/a'e^{-\sqrt{2}/2}} \tag{6}
$$

$$
c'_1 = \frac{\sqrt{2}}{2}\left(\frac{1}{1+1/a'e^{\sqrt{2}/2}} - \frac{1}{1+1/a'e^{-\sqrt{2}/2}}\right)
$$

$$
l(x) \approx \left(\frac{c'_0}{2} + c'_1(-1-dx^2)\right)\beta' x = \left(\frac{c'_0}{2} - c'_1\right)\beta' x - c'_1 d\beta' x^3
$$

$$
\begin{aligned}
l''(x) &= -6c'_1 d\beta' x \\
&= -3\sqrt{2}\left(\frac{a'}{a'+e^{\sqrt{2}/2}} - \frac{a'e^{\sqrt{2}/2}}{a'e^{\sqrt{2}/2}+1}\right)d\beta' x \\
&= -3\sqrt{2}\frac{a' - a'e^{\sqrt{2}}}{(1+a'e^{\sqrt{2}/2})(a'+e^{\sqrt{2}/2})}d\beta' x
\end{aligned}
\tag{7}
$$

Suppose $m(x) = E[Y|X=x]''$, then
$m(x) = h''(x) + l''(x) = 3\sqrt{2}\frac{a\prime - a\prime e^{\sqrt{2}}}{(1+a\prime e^{\sqrt{2}/2})(a\prime + e^{\sqrt{2}/2})}d(\beta - \beta')x$.
Since $a' = \frac{\sigma_X p_2}{\sigma'_X p_1}e$ and $p_1 = 1 - p_2$, p_2 increasing is equivalent to a' increasing. So to find how the concavity changes before and during crises, we check how the concavity changes when a' goes up.

We use $\int_{-\infty}^{+\infty} -\min(m(x),0)dx$ to measure the concavity of $E[Y|X=x]$. From our toy model's assumptions, $\beta' > \beta$, and considering $d = \frac{\sigma'^2_X - \sigma^2_X}{2\sigma^2_X \sigma'^2_X} > 0$, we have $sign(m(x)) = sign(x)$, $m(-x) = -m(x)$. Then $\int_{-\infty}^{+\infty} -\min(m(x),0)dx = \int_0^{+\infty} m(x)dx$.

Considering the domains of returns in the market, in practice we limit this integral to a finite interval [m–$k\sigma$, m + $k\sigma$], where m and σ are the mean and the standard deviation of X, and k is a multiplier to be chosen later.

Now we analyze how $m(x)$ changes when a' goes up for $x \geq 0$.

$$\begin{aligned}\frac{\partial m}{\partial a'} &= 3\sqrt{2}dx(\beta' - \beta)\partial\left(\frac{a'e^{\sqrt{2}} - a'}{(1 + a'e^{\sqrt{2}/2})(a' + e^{\sqrt{2}/2})}\right)/\partial a' \\ &= 3\sqrt{2}dx(\beta' - \beta)\frac{(1 - a'^2)(e^{3\sqrt{2}/2} - e^{\sqrt{2}/2})}{[(1 + a'e^{\sqrt{2}/2})(a' + e^{\sqrt{2}/2})]^2}\end{aligned} \quad (8)$$

From our deviation before, $a' = \frac{\sigma_X p_2}{\sigma'_X p_1}e$. When $a' = 1$, we have $\frac{\partial m}{\partial a'} = 0$, and $m(x, a')$ gets maximum for fixed $x \geq 0$. Under normal market conditions, $a' < 1$. So before a crisi, p_2 increases and a' gets closer to 1. And then $m(x)$ increases for any $x > 0$. So the concavity of fitted polynomial $E[Y|X = x]$, $\int_{-\infty}^{+\infty} - \min(m(x), 0)dx = \int_0^{+\infty} m\,(x)dx$ increases.

There is a pitfall in our model. During a crisis, if p_2 continues increasing and a' gets larger than 1, the concavity may decrease. So with our model, we can only predict when a crisis starts, but not when it ends.

3. POLYMODEL

In this paper, we use Polymodel to fit S&P 500, which can represent the U.S. market. Measuring the concavity from Polymodel, we get the concavity of the market.

Factor model is one of the most commonly used model in financial industry. Factor models includes single-factor models and multi-factor models. A single factor model has fewer coefficients, and is easier to calibrate with limited amount of historical data. Meanwhile, a multi-factor model may capture more uncertainty of Y with more factors.

Polymodel combines the advantages of single factor model and multi-factor model. It is actually a collection of hundreds of polynomial single factor models. The intuition behind the idea of Polymodel is that a collection of single factor models of a given random variable (or possibly models with a reduced number of dimensions) contains exactly as much information as one big multi-factor model with the same number of factors. [11]

Polymodel with n factors has form:

$$\begin{cases} Y = f_1(X_1) + \epsilon_1 \\ Y = f_2(X_2) + \epsilon_2 \\ \vdots \\ Y = f_n(X_n) + \epsilon_n \end{cases} \quad (9)$$

3.1 Base Functions

In Polymodel, $f_i(X_i)$ is a polynomial function of X_i. It is a general linear function $f_i(X_i) = \sum_{k=0}^{K} \beta_{ik} U_k(X_i)$, $U_k(X_i)$'s are some orthogonal polynomial sequence of X_i with kth-degree of order. In this paper, $U_k(X_i)$ is the second kind Chebyshev polynomials with degree k. $U_k(X_i)$s are orthogonal with respect to weighting function $\sqrt{1-x^2}$ on $[-1,1]$.

To apply the second kind Chebyshev polynomials, we scale our returns data in experiments. In the experiments of this paper, we normalize the data.

3.2 Coefficients Estimation

For each single equation in Polymodel, we have $f_i(X_i) = E[Y|X_i]$. We can optimize the coefficients β_{ik}'s for each X_i. To reduce multicolinearity, we also apply ridge regression in model fitting.

Our ridge regression objective function is:

$$minimize \sum_{j=1}^{m} (y_j - Z_j^T \beta)^2 s.t. \beta^T W \beta \leq c \tag{10}$$

m is the number of training data points. $\beta = (\beta_0, \beta_1, \dots, \beta_K)$. $Z_j = (U_0(x_{ij}), U_1(x_{ij}), \dots, U_K(x_{ij}))$. W is the penalty weight matrix. Unlike traditional ridge regression, in this paper, we put more constraints on βs of higher degrees, and set $W = diag([0,1,4, \dots, K^2])$.

Applying Lagrange optimization method, we have,

$$\hat{\beta}_{ridge,\lambda} = (Z^T Z + \lambda W)^{-1} Z^T y \tag{11}$$

λ controls the amount of regularization. We apply K-fold cross validation method, and find that as long as λ is larger than 0, errors are pretty similar with different λs. To make Polymodel stable, we just set $\lambda = 0.1$ in our experiments.

Other alternative methods to control coefficients would be the LASSO [12], or the Elastic-Net [13] approaches. However, these two methods include a penalty of l^1 Norm, that may make higher order β be 0 and lead to discard of much concavity. We want to measure the concavity in this paper, so ridge regression is the best option.

3.3 Factor Importance

After solving every single factor model in Polymodel, we want to combine all information we get from them. We do that with the importance of factors.

In Polymodel, we usually use hundreds of factors to fit Y. Some factors may fit Y better than others, in which case we assume those factors are more important. However, we do not know which factors are important before building the model. And the importance of factors changes while time goes by. So we measure the importance of all factors at every time point.

We measure factor importance using the following statistics in this paper.

$\boldsymbol{R^2}$: R^2 is the coefficient of determination. It is a statistical measure of the proportion of the variance in Y that is predictable from Xs. Larger R^2 means better fitting.

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST} \tag{12}$$

where $SSR = \sum_{j=1}^{m}(\hat{y}_j - \bar{y})^2$ is the sum of squares of regression, $SSE = \sum_{j=1}^{m}(y_j - \hat{y}_j)^2$ is the sum of squares of errors, $SST = \sum_{j=1}^{m}(y_j - \bar{y})^2 = SSR + SSE$ is the total sum of squares. $\hat{y}_j$ is the fitted value with x_j.

$\boldsymbol{Effective\ R^2}$: We define effective R^2 in this paper. The intuition is from effective transfer entropy in information flow. [14]

It would be great if we could have data size as large as possible in order to find the relationship between Y and X. However, in practice we only have limited data points, especially the market patterns change over time, and we want to find the relationship of Y and X with recent historical data. A potential pitfall is that the data points are too few that the past observations are misinterpreted, and R^2 indicates stronger relationship between Y and X than the reality.

This situation is similar to the situation in measuring information transfer entropy, where people use effective transfer entropy to solve the problem. Considering that, we define a new statistic: effective R^2.

$$effectiveR^2 = R^2 - R^2_{shuffle} \tag{13}$$

We shuffle the samples of Y or X for many times, at the ith time, we get a $R^2_{i-shuffle}$with shuffled data. $R^2_{shuffle}$ is the mean of $R^2_{i-shuffle}$s of all the times.

$\boldsymbol{-ln(p-value)}$**:** p-value is the probability that we can get higher R^2 between Y and X. It is calculated as the percentage of all $R^2_{i-shuffle}$s that are larger than R^2. The closer p-value to 0, the more important the corresponding factor is. We measure the importance of factors using $-ln(p-value)$. The higher $-ln(p-value)$, the more important a factor is.

F-statistic: F-statistic is the F-statistic in Analysis of Variance Table in multiple regression.

$$F = \frac{MSR}{MSE} = \frac{\sum_{j=1}^{m}(\hat{y}_j - \bar{y})^2}{\sum_{j=1}^{m}(y_j - \hat{y}_i)^2}\frac{m-(K+1)}{K} \tag{14}$$

m is the size of training data set, K is the highest degree in our Polymodel. R^2 is the coefficient of determination. The higher the F, the more significant the relationship between independent variables and dependent variables is.

3.4 Measure the Concavity from Polymodel

As our toy model built in Section 2.1, we integrate the negative part of second derivative to measure the concavity of every single factor model in Polymodel. We normalize X_i to X'_i in Polymodel. Suppose the sample mean of X_i is m_i, and sample standard deviation is std_i. Then $X'_i = \frac{X_i - m_i}{std_i}$.

$Y = f(X_i) = f'(X_i')$, then,

$$Conc_i = Conc(f_i(x)) = \int_{-\infty}^{+\infty} -min(f_i''(x),0)dx \tag{15}$$

$$\frac{\partial^2 Y}{\partial X^2} = \partial(\frac{\partial Y}{\partial X})/\partial X = \frac{\partial\left(\frac{\partial Y}{\partial X'}\frac{\partial X'}{\partial X}\right)}{\partial X'} \cdot \frac{\partial X'}{\partial X} = \frac{1}{std_i^2}\frac{\partial^2 Y}{\partial X'^2} \tag{16}$$

The integral is from $-\infty$ to $+\infty$. However, in practice, we can only calculate the integral form $m_i - std_i$ to $m_i + std_i$, and the result is close to when we assume the relationship between Y and X beyond that scope is linear. Since returns are usually highly correlated during market collapse, that assumption makes sense. Then,

$$\begin{aligned} Conc_i = Conc(f_i(x)) \quad &= \int_{m_i - std_i}^{m_i + std_i} -min(f_i''(x),0)dx \\ &= \int_{-1}^{1} -\frac{1}{std_i} min((f_i'(x'))'', 0)dx' \end{aligned} \tag{17}$$

By the above steps, we convert the concavity to a simple form in calculation.

For every single factor model in Polymodel, we get concavity $Conc_i$. Then,

$$Conc(Y) = \sum_{i=1}^{n} w_i \, Conc_i \quad w_i = \frac{M_i}{\sum_{i=1}^{n} M_i} \tag{18}$$

M_i is some factor importance statistic of $Y = f_i(X_i) + \epsilon_i$.

4. EMPIRICAL CRISIS INDICATOR

4.1 Data Process

We set the monthly return of S&P 500 Index as Y in the Polymodel. Our factor pool contains about 180 factors, including global equity indices, currency indices, bond and yield indices and commodity indices. Sample factors are shown in Exhibit 1.

Our data is from January 1990 to June 2020. Since some indices only exists during part of the whole period, we use rolling data in our experiments. At the end of every month, we only use indices that have existed for more than 36 months.

Exhibit 1: Sample Indices Factors

Category	Ticker	Description
Equity	SHCOMP Index	SSE Composite Index
Equity	STI Index	Singapore Stock Market Index
Equity	DAX Index	30 Major German Stocks Index
Currency	DXY Index	US Dollar Index
Currency	USDCNY Curncy	USD to Chinese Yuan Exchange
Currency	USDJPY Curncy	USD to Japanese Yen Exchange
Bound & Yield	IRX	US 13 Week Treasury Bill Yield
Bound & Yield	USGG3M Index	US Government 3-Month Bond Yield
Bound & Yield	USGG5YR Index	US Government 5-year Bond Yield
Commodity	BCOMCN Index	Corn
Commodity	BCOMAG Index	Agriculture
Commodity	BCOMNG Index	Natural Gas

Note: These factors are the same as in [15].

4.2 Results

Exhibit 2 and Exhibit 3 shows the concavity versus price of S&P 500. We did experiments with different factor importance measures and got similar results. From the pictures, we find that, no matter which measure we use, the concavity increases before crises, and decrease during or after crisis.

Exhibit 2: Concavity with Different Factor Importance Measures (I)

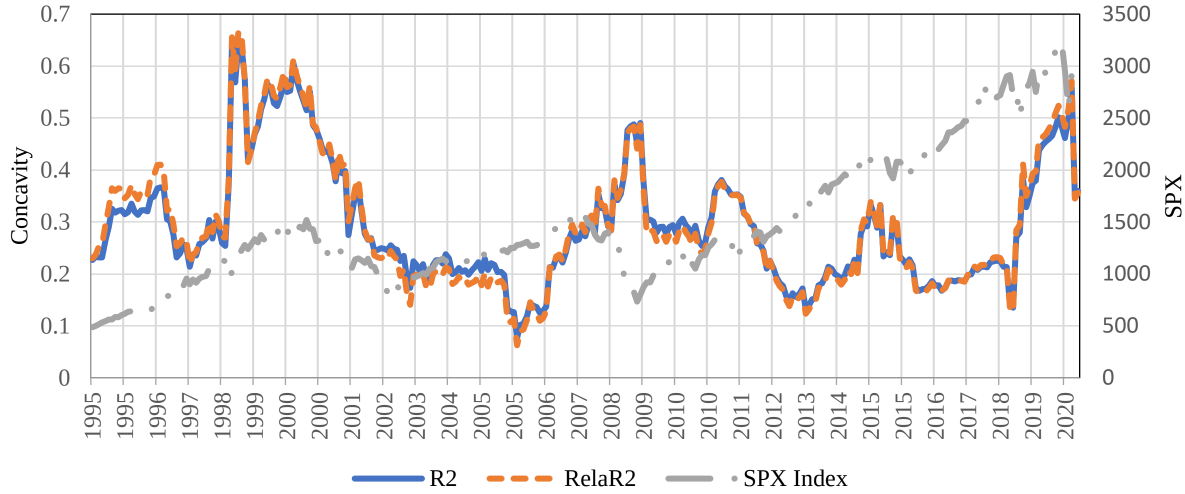
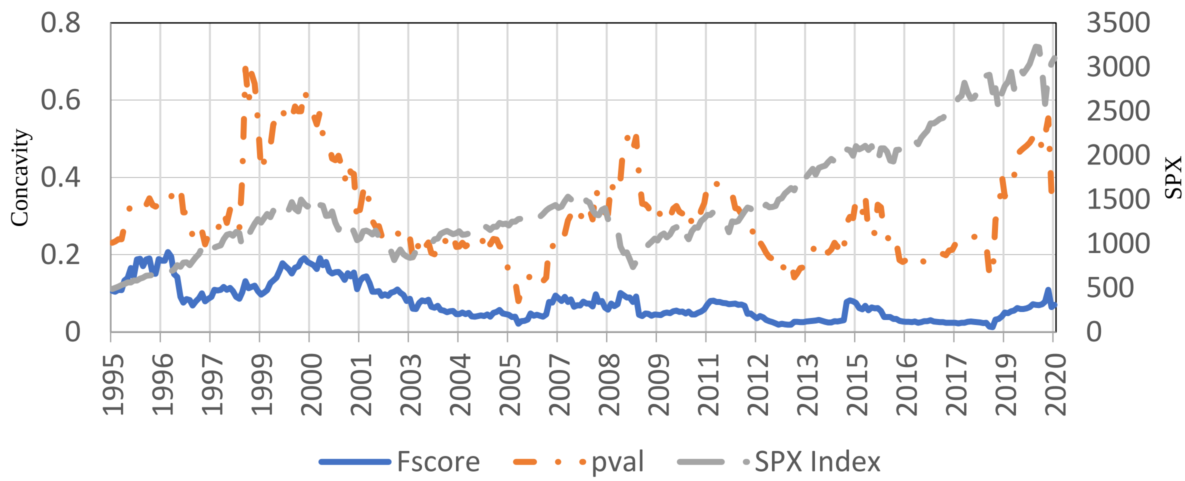
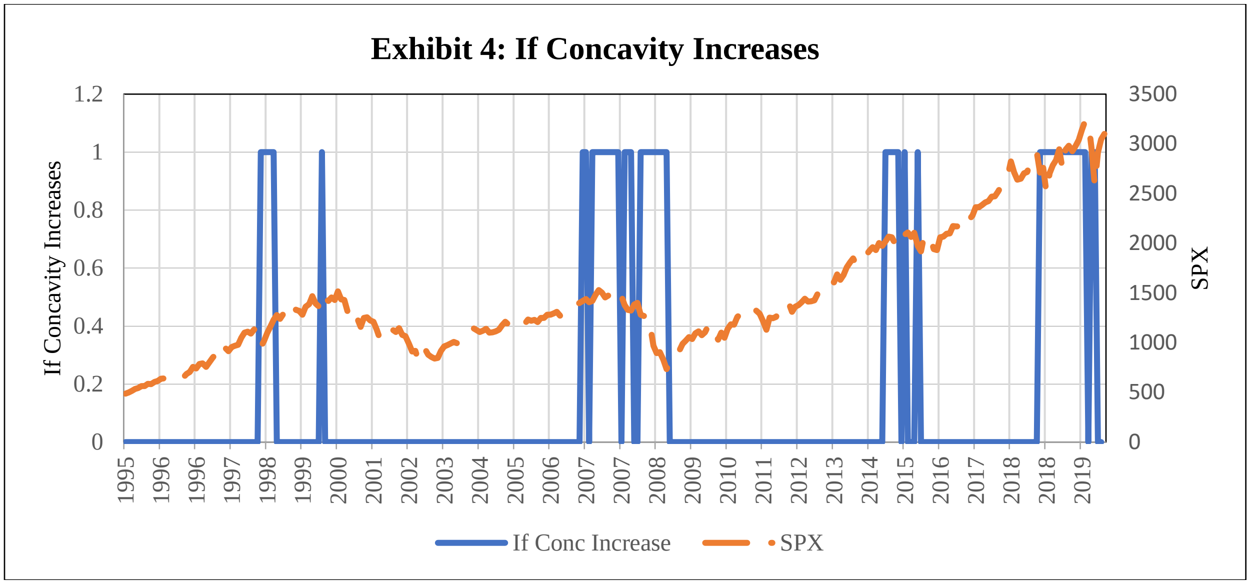


Exhibit 3: Concavity with Different Factor Importance Measures (II)



To make the results more clear, we set some criteria to judge if the concavity increases at a specific month. We simply compare the current concavity with its 90th percentile during the past 3 years. We build 4 different "importance measures" that combine the concavity of S&P 500 with respect to each factor. We use ensemble result here, which means that we only consider the global market concavity to be larger than usual if the concavity got with at least 3 of our importance measures are above their 90th percentile. The result shown in Exhibit 4 evidence the fact that, before crises, the global concavity increases.



4.3 Different Frequency Data Results

We also check if concavity as crisis indicator still works using different frequency data. We did experiments with bi-weekly data. At the end of every two-weeks period, we built a Polymodel of S&P 500 using the past 3 years of data, and measured the concavity. The concavity results are shown in Exhibit 5 and Exhibit 6.

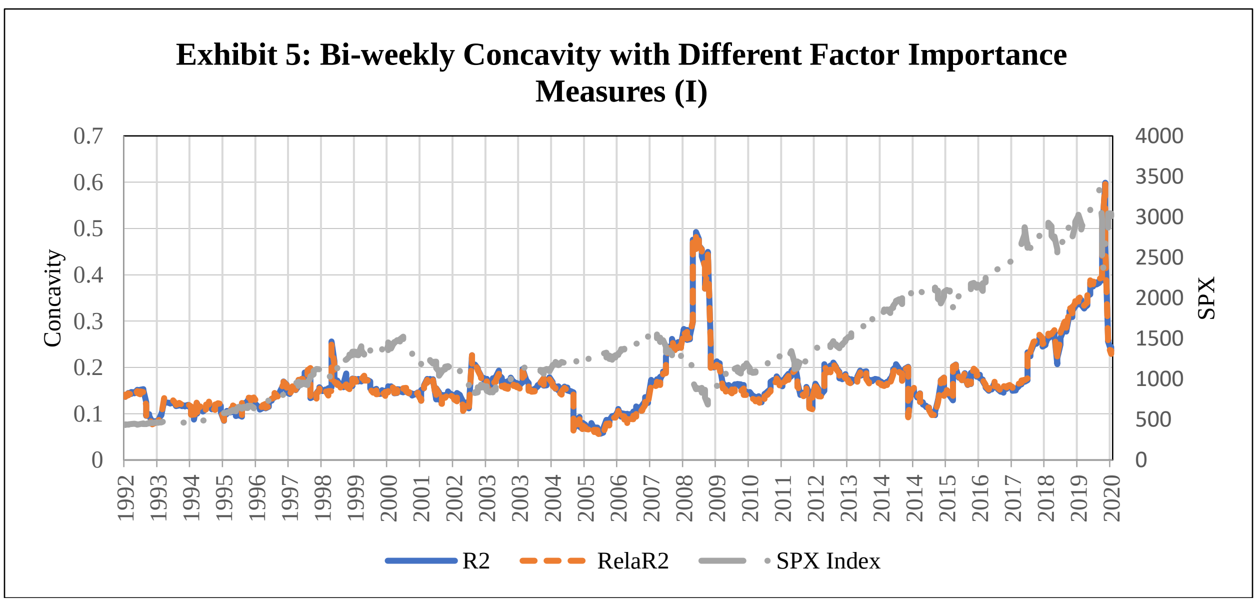
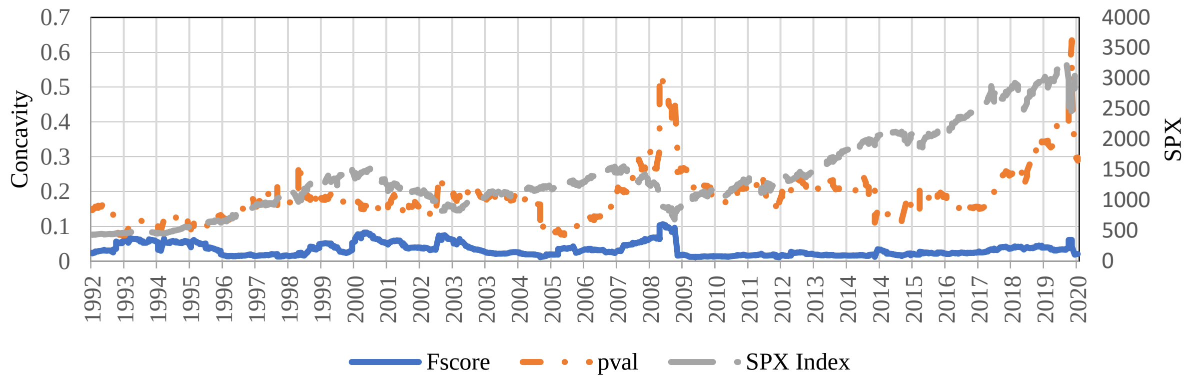
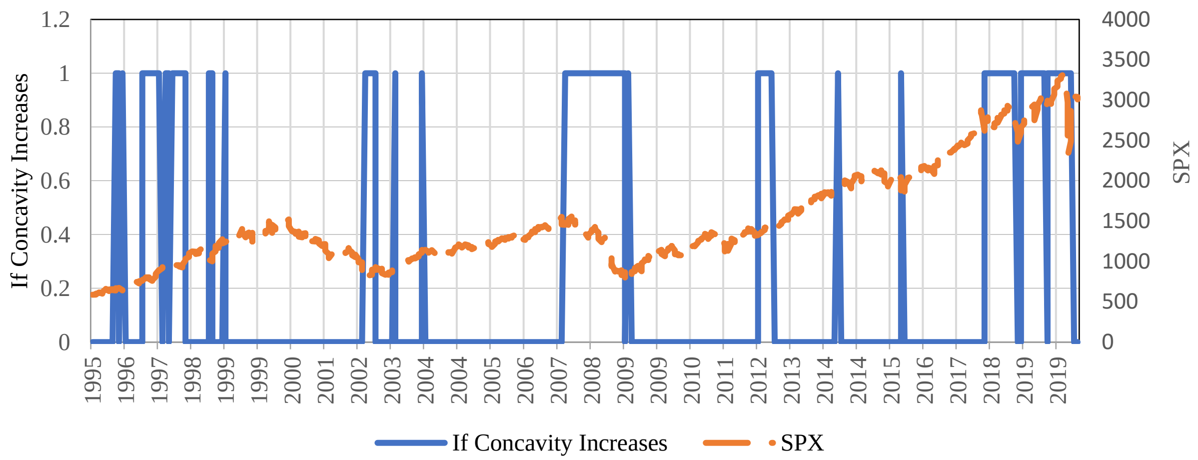


Exhibit 6: Bi-weekly Concavity with Different Factor Importance Measures (II)



We compare the concavity of the market from four importance measures with their 90th percentile over the past 3 years, and get the ensembled result in Exhibit 7. The concavity increases before crises. Please note that the concavity results are got at the beginning of each month, while SPX data is got at the end of each month.

Exhibit 7: If Concavity Increases (Bi-weekly)



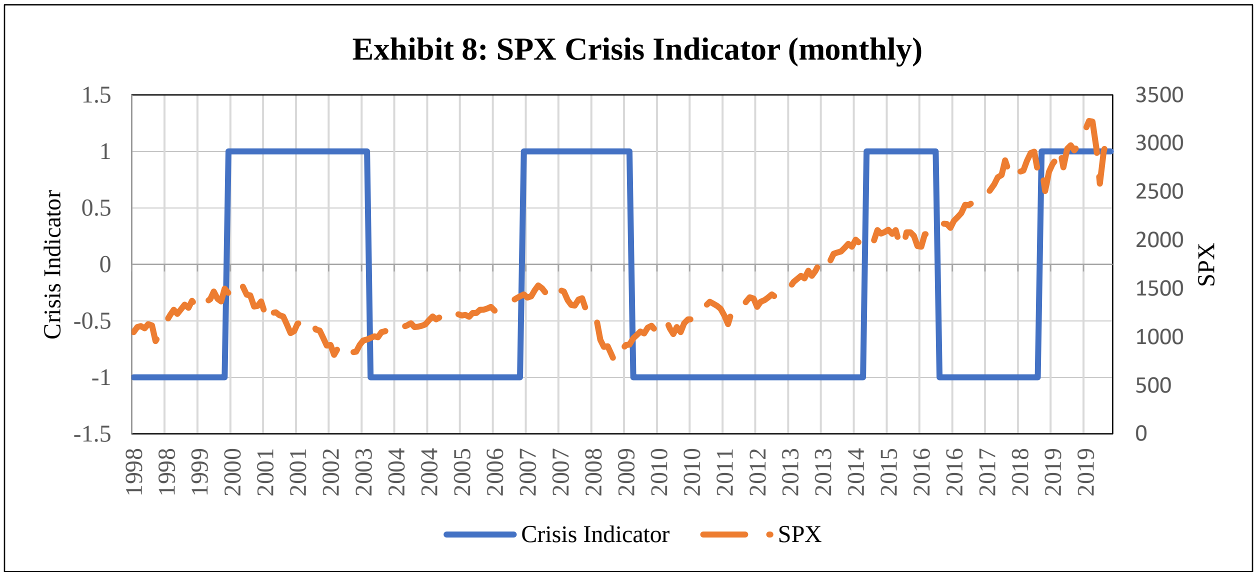
5. TRADING STRATEGY

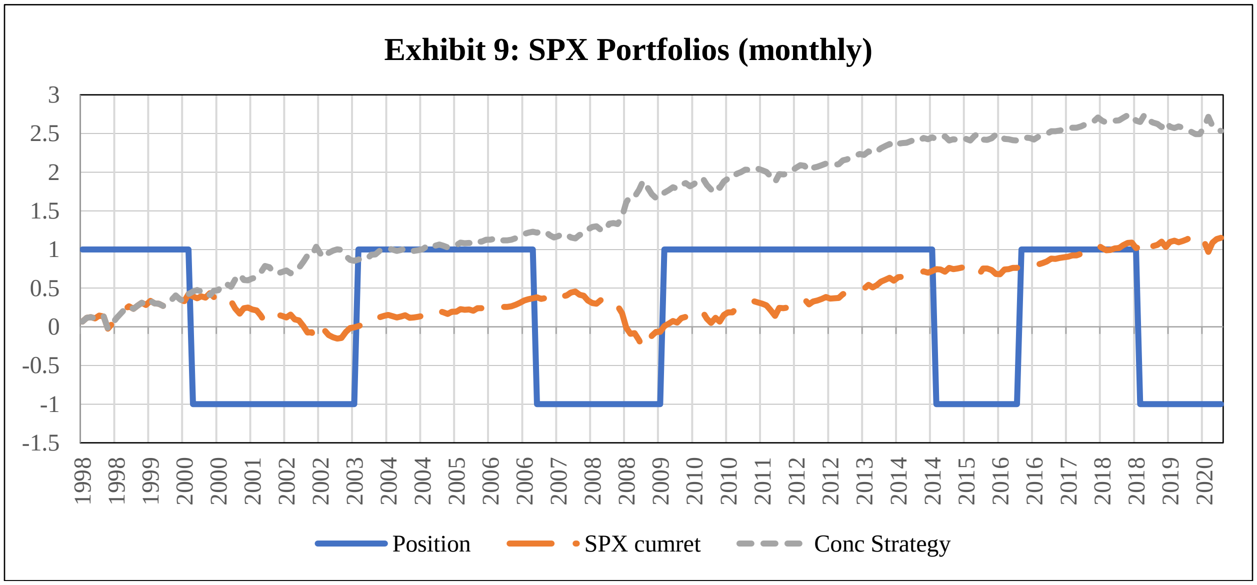
In this section, we build trading strategies based on monthly results. Let us return to our toy model from Section 2.1. From the deviation, we know that before and during the beginning of a crisis, the concavity increases. However, if the crises continue for some time, the concavity may decrease.

Considering that, we built an indicator with a start point based on concavity, but end point based on other statistics. We set the start point of a market crisis when the concavity increases, the end point when the indicator indicates in a crisis last month, but S&P 500 price increases for the past 4 consecutive months.

As long as we get the crisis indicator, we can build a trading strategy: we hold a long position of S&P 500 when the crisis indicator is negative, and a short position when the indicator is positive. To avoid huge losses, we also set a stop-loss criterion: from the last time we change the position, if the loss from peak cumulative return is more than 10%, we close the current position for 3 months. For example, if we hold a long position of S&P 500, but lose 10% in cumulative return from the peak since we hold the long position, we close the long position for the next 3 months. Meanwhile, we may short S&P 500 in the next three months if the crisis indicator indicates a crisis.

Exhibit 8 shows the crisis indicator we get at the beginning of every month. Exhibit 9 shows our positions in this strategy, and our portfolio. The portfolio's statistics are shown in Exhibit 10.





In our experiments, we assume the risk free rate is 0. It does not influence the Sharpe ratio much since the risk free rate in the market is usually close to 0.

Exhibit 10: Strategies Statistics (S&P 500 monthly)						
yearly	**log-r (%)**	**std**	**Sharpe**	**r (%)**	**MDD (%)**	**Calmar**
S&P 500	5.14	0.155	0.332	5.15	52.6	0.098
Concavity	11.30	0.152	0.744	11.35	22.4	0.506

In Exhibit 10, we find that our strategy using our concavity-based crisis indicator work much better than the S&P 500 index. Our strategy reduces losses signficantly, while displaying appreciable profits during market crises. The strategy's return, Sharpe ratio, Calmar ratio (annual return over maximum drawdown) are improved, and the maximum drawdown (MDD) is lower than that of the S&P 500. In addition, the average holding period is rather long (a couple of years), which makes trading cost very low. The volatility is close to that of the S&P 500. That is because our strategy simply holds a long or short position of S&P 500, which cannot change the standard deviation much.

Other strategies can be devised based on the concavity change. For example, one can use options to protect their portfolio from large losses as soon as the concavity in the market indicates an increase in the propoability of occurrence of acrisis.

6. CONCLUSIONS

In this paper, we show how to predict the crisis risk using market concavity, measured using a Polymodel methodology. From these indicators, we are able to build a trading strategy that turns losses into profits during crises. For the computation of our concavity measures, we are led to define a new factor importance measure in Polymodel, the "effective R^2".

From a toy model, we could conclude that before crises, the concavity between returns should increase. We devised a *Polymodel* technique to measure the concavity of indices and applied it to monthly and bi-weekly returns of the S&P 500 Index. The experiments results show that that conclusion remains robust to a change of frequency. The trading strategy we were able to build, based on the monthly crisis indicator, generates yearly returns up to 11.30% and yearly Sharpe ratio 0.744.

Our plans for the future are, first, to determine an indicator that a crisis has ended. We also consider testing new sets of factors for the polymodal, aside from traditional indices, for instance using individual or aggregated stock returns, in order to improve the accuracy of the crisis prediction.

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