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Tempered Stable Processes with Time Varying Exponential Tails

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In this paper, we introduce a new time series model having a stochastic exponential tail. This model is constructed based on the Normal Tempered Stable distribution with a time-varying parameter. The model captures the stochastic exponential tail, which generates the volatility smile effect and volatility term structure in option pricing. Moreover, the model describes the time-varying volatility of volatility. We empirically show the stochastic skewness and stochastic kurtosis by applying the model to analyze S&P 500 index return data. We present the Monte-Carlo simulation technique for the parameter calibration of the model for the S&P 500 option prices. We can see that the stochastic exponential tail makes the model better to analyze the market option prices by the calibration.

Keywords: Option pricing, Stochastic exponential tail, Volatility of volatility. Normal tempered stable distribution, Lévy process

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1. Introduction

Since Black and Scholes (1973) has found a no-arbitrage pricing model for European options, the Black-Scholes model became a standard method to explain the option market. However, the model does not describe option markets' features, including the volatility smile effect and fat-tailed events. Many alternative methods are introduced to overcome the drawback of the model. There are two major methods to overcome the drawbacks: (1) generalizing the Black-Scholes model with Lévy process and (2) applying time-varying stochastic volatility on the Black-Scholes model.

The generalized Black-Scholes model with the Lévy process has first applied to the option pricing problem in Bouchaud and Sornette (1994) and Bouchaud *et al.* (1997). After then, many Lévy process models were studied in the literature. For instance, the tempered stable process, which is a subclass of Lévy process has been popularly used as an option pricing model (see Barndorff-Nielsen and Levendorskii (2001), Barndorff-Nielsen and Shephard (2001), and Carr *et al.* (2002)), since the class of tempered stable processes are semi-martingale and has exponential tails which are fatter than Gaussian distribution. Moreover, since the tempered stable model's tails can be asymmetric, the tempered stable option price model explains the volatility smile effect. However, Since Lévy process is Markovian and has independent and stationary increments, the class of tem-

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pered stable models does not capture the volatility clustering effect, stochastic skewness, stochastic kurtosis, and volatility of volatility (vol-of-vol).

Managing volatility and vol-of-vol are important issues in portfolio and risk management and derivative pricing. Since they are not directly observed in the market, VIX index (CBOE (2009)) and VVIX index (CBOE (2012)) are provided for measuring the volatility and vol-of-vol of U.S. stock market, respectively. Academically, ARCH and GARCH models by Engle (1982) and Bollerslev (1986). The implied volatility extracted from the Black-Scholes model (Black and Scholes (1973)) has been popularly used to observe the volatility.

Applying the ARMA-GARCH model to empirical daily log-returns of a stock or an index, we can see that the residual distribution still has fat-tails and asymmetricity (see Kim *et al.* (2008, 2011)). In order to capture those the fat-tails and skewness of the residual distribution, ARMA-GARCH model with the standard normal tempered stable innovation distribution (ARMA-GARCH-NTS model) was studied in risk management and portfolio management in many literatures including Kim (2015), Anand *et al.* (2016, 2017) and Kurosaki and Kim (2018). The normal tempered stable (NTS) distribution was presented in finance by Barndorff-Nielsen and Levendorskii (2001) and Barndorff-Nielsen and Shephard (2001) to describe the fat-tail and skewness of asset returns. The standard NTS (stdNTS) distribution is a special case of the NTS distribution with zero mean and unit variance (See Rachev *et al.* (2011)).

The volatility clustering and fat-tailed asymmetric distribution have been studied in option pricing in literature. The Lévy process model, stochastic volatility model, and GARCH model were introduced to overcome the drawback of Black-Scholes option pricing model. For instance, the Lévy stable model was applied to the option pricing in Hurst *et al.* (1999) and Carr and Wu (2003). The tempered stable option pricing models were discussed in Boyarchenko and Levendorskiĭ (2002) and Carr *et al.* (2002). Stochastic volatility model was applied to option pricing in Heston (1993), and stochastic volatility model with the Lévy driving process has been studied in Carr *et al.* (2003). The discrete-time volatility clustering effect was considered for option pricing by taking GARCH model in Duan (1995). GARCH option pricing model with non-Gaussian tempered stable innovation was studied by Kim *et al.* (2010) and regime-switching tempered stable model were applied to the option pricing in Kim *et al.* (2012). The skewness and kurtosis were used in addition to volatility for option pricing in Aboura and Maillard (2016). Moreover, Lévy process model with long-range dependence was presented in Kim *et al.* (2019).

While the stochastic volatility and volatility clustering were studied, the term structure of vol-of-vol was studied for VIX and VVIX derivatives pricing. For example, the class of Lévy Ornstein Uhlenbeck process is used for modeling vol-of-vol in Mencia and Sentana (2013) and the Heston style term structure of vol-of-vol has been presented in Huang *et al.* (2018), and Branger *et al.* (2018). Also, Fouque and Saporito (2018) considered the Heston style volatility model for the volatility together with the dependence feature between VIX and S&P 500 index.

In this paper, we construct a new market model that has stochastic exponential tails. Using the stochastic exponential tails, the new model can capture more stochastic properties observed in the market, including stochastic skewness and kurtosis, and volatility of volatility (vol-of-vol). First, we will discuss two empirical properties of skewness and excess kurtosis: (1) the residual distribution of S&P 500 index daily return has negative skewness and large excess kurtosis. Moreover, the absolute value of skewness is increasing then the excess kurtosis is rising together. (2) Skewness and excess kurtosis of S&P 500 index daily return distribution are not constant but time-varying. Then we will present a new advanced model named the Stochastic Tail NTS (StoT-NTS) model to describe those two properties. In order to construct the model, we take the ARMA-GARCH-NTS model and apply a simple time series model to one shape parameter of stdNTS distribution. After constructing the model, a parameter estimation method for the StoT-NTS model will be provided. Using the model, one can capture the time-varying vol-of-vol on stock or index return process. After the model construction, we apply the model to option pricing. We discuss the Monte-Carlo simulation algorithm for European option pricing on the StoT-NTS model. To verify the model's performance, we calibrate the model's parameters using the S&P 500 index option prices. As mentioned in the previous paragraph, Aboura and Maillard (2016) considers the skewness and excess kurtosis in option pricing, while we consider a parametric model with stochastic skewness and stochastic kurtosis in option pricing in this paper. The StoT-NTS option pricing model can extract the structure of invisible time-varying vol-of-vol in the market option prices.

The remainder of this paper is organized as follows. The NTS distribution is discussed in Section 2. In Section 3, we present the stochastic properties of skewness and excess kurtosis of the residual distribution for ARMA-GARCH model and empirical study using the S&P 500 index daily return data. The StoT-NTS model is constructed in this section and shows the model has time-varying vol-of-vol. The option pricing model on the StoT-NTS model is discussed in Section 4. The Monte-Caro algorithm and model calibration are also provided in this section. Finally, Section 5 concludes.

2. Normal Tempered Stable Distribution

Let $\alpha \in (0, 2)$, $\theta, \gamma > 0$, and $\mu, \beta \in \mathbb{R}$. Let $\mathcal{T}$ be a positive random variable whose characteristic function $\phi_{\mathcal{T}}$ is equal to

$$\phi_{\mathcal{T}}(u) = \exp\left(-\frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left((\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right). \tag{1}$$

The random variable $\mathcal{T}$ is referred to as *Tempered Stable Subordinator*. The *normal tempered stable* (NTS) random variable X with parameters $(\alpha, \theta, \beta, \gamma, \mu)$ is defined as

$$X = \mu - \beta + \beta\mathcal{T} + \gamma\sqrt{\mathcal{T}}W, \tag{2}$$

where $W \sim N(0, 1)$ is independent of $\mathcal{T}$, and we denote $X \sim \text{NTS}(\alpha, \theta, \beta, \gamma, \mu)$. The characteristic function (Ch.F) of X is given by

$$\begin{aligned}\phi_{NTS}(u) &= E[e^{iuX}]\\ &= \exp\left((\mu-\beta)iu - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - i\beta u + \frac{\gamma^2u^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right).\end{aligned}$$

The first four moments of X are as follows:

- Mean: $E[X] = \mu$
- Variance: $\text{var}(X) = \gamma^2 + \beta^2\left(\frac{2-\alpha}{2\theta}\right)$
- skewness: $\text{S}(X) = \frac{\beta\,(2-\alpha)\,\left(6\,\gamma^2\,\theta - \alpha\beta^2 + 4\beta^2\right)}{\sqrt{2\theta}\,\left(2\,\gamma^2\,\theta - \alpha\beta^2 + 2\beta^2\right)^{3/2}}$
- Excess kurtosis: $\text{K}(X) = \frac{(2-\alpha)\,\left(\alpha^2\beta^4 - 10\,\alpha\beta^4 - 12\,\alpha\beta^2\,\gamma^2\,\theta + 24\beta^4 + 48\beta^2\,\gamma^2\,\theta + 12\,\gamma^4\,\theta^2\right)}{2\,\theta\,\left(2\,\gamma^2\,\theta - \alpha\beta^2 + 2\beta^2\right)^2}$

Hence, if $\mu = 0$ and $\gamma = \sqrt{1 - \beta^2\left(\frac{2-\alpha}{2\theta}\right)}$ with $|\beta| < \sqrt{\frac{2\theta}{2-\alpha}}$ then $\epsilon \sim \text{NTS}(\alpha, \theta, \beta, \gamma, \mu)$ has zero mean and unit variance. Put $\beta = B\sqrt{\frac{2\theta}{2-\alpha}}$ for $B \in (-1, 1)$, then $|\beta| < \sqrt{\frac{2\theta}{2-\alpha}}$ and $\gamma = \sqrt{1 - B^2}$.

Then the Ch.F of ϵ equals to

$$\begin{aligned}\phi_\epsilon(u) &= E[e^{iu\epsilon}] \\ &= \exp\left(-iuB\sqrt{\frac{2\theta}{2-\alpha}} - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - iuB\sqrt{\frac{2\theta}{2-\alpha}} + \frac{u^2}{2}\left(1-B^2\right)\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right)\end{aligned}$$

In this case ϵ is referred to as the *standard NTS* random variable with parameters $(\alpha, \theta; B)$, and we denote $\epsilon \sim \text{stdNTS}(\alpha, \theta; B)$.[1] The Ch.F is denoted by $\phi_{stdNTS}(u; \alpha, \theta; B) = \phi_\epsilon(u)$. For ϵ, we have

$$S(\epsilon) = \sqrt{\frac{2-\alpha}{2\theta}}B\left(3(1-B^2) + \frac{4-\alpha}{2-\alpha}B^2\right) \tag{3}$$

and

$$K(\epsilon) = \frac{(2-\alpha)}{2\theta}\left((\alpha-4)(\alpha-6)\left(\frac{B^2}{2-\alpha}\right)^2 + \left((24-6\alpha)\left(\frac{B^2}{2-\alpha}\right) + 3(1-B^2)\right)(1-B^2)\right). \tag{4}$$

Suppose that α and θ are fixed then we have a function

$$B \mapsto (\mathrm{S}(\epsilon), \mathrm{K}(\epsilon)), \quad \theta > 0.$$

We can easily check the following facts:

- if $B = 0$

$$\mathrm{S}(\epsilon) = 0 \quad \text{and} \quad \mathrm{K}(\epsilon) = \frac{3}{2\theta}(2-\alpha).$$

- if $B = \pm 1$ then $\gamma = \sqrt{1-B^2} = 0$, and hence

$$S(\epsilon) = \frac{\pm(4-\alpha)}{\sqrt{2\theta(2-\alpha)}} \quad \text{and} \quad K(\epsilon) = \frac{(\alpha-4)(\alpha-6)}{2\theta(2-\alpha)}.$$

For example,

- if $\alpha = 1.8$ and $\theta = 1.5$, then $\mathrm{S}(\epsilon) \in [-2.8402, 2.8402]$ and $\mathrm{K}(\epsilon) \in [0.2, 15.4]$.
- if $\alpha = 0.8$ and $\theta = 3$, then $\mathrm{S}(\epsilon) \in [-1.1925, 1.1925]$ and $\mathrm{K}(\epsilon) \in [0.6, 2.3111]$.

Other example cases of the function are presented in the Figure 1. The points of $(\mathrm{S}(\epsilon), \mathrm{K}(\epsilon))$ are smoothly connected parabolic curve for $B \in [-1, 1]$.

[1] The standard NTS distribution is defined by the NTS distribution with $\mu = 0$ and $\gamma = \sqrt{1-\beta^2\left(\frac{2-\alpha}{2\theta}\right)}$ under the condition that $|\beta| < \sqrt{\frac{2\theta}{2-\alpha}}$ and denoted to stdNTS(α, θ, β), in many literature including Kim and Kim (2018), Anand *et al.* (2016), Anand *et al.* (2017), and Kim *et al.* (2015). In this paper, we change the parameterization for the convenience.

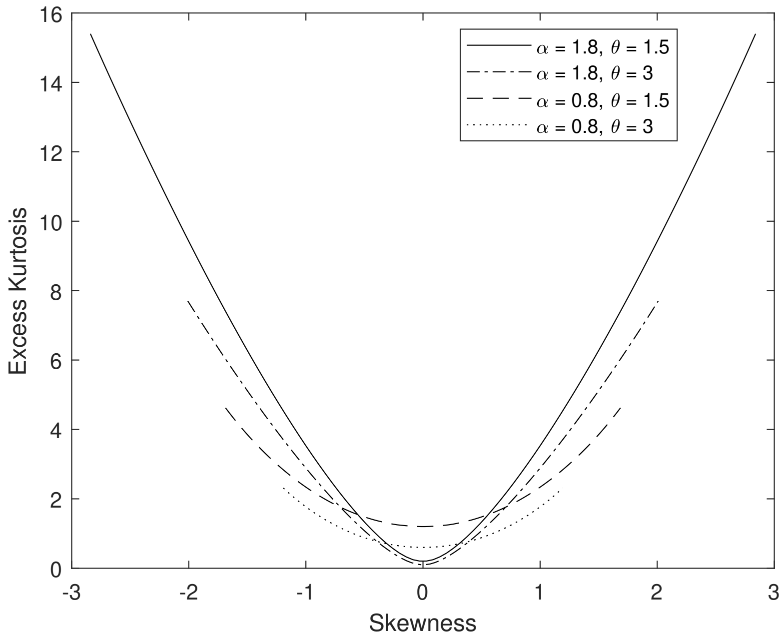


Figure 1. Graph of skewness to Excess kurtosis for $\epsilon \sim \text{stdNTS}(\alpha, \theta; B)$ with $(\alpha, \theta) \in \{(1.8, 1.5), (1.8, 3), (0.8, 1.5), (0.8, 3)\}$ and $B \in [-1, 1]$.

3. ARMA-GARCH-NTS Model with Stochastic Parameter B

Taking the ARMA(1,1)-GARCH(1,1) model as

$$\begin{cases} y_{t+1} = c + ay_t + b\sigma_t\epsilon_t + \sigma_{t+1}\epsilon_{t+1} \\ \sigma_{t+1}^2 = \kappa + \xi\sigma_t^2\epsilon_t^2 + \zeta\sigma_t^2 \end{cases},$$

we assume that $\epsilon_t \sim \text{stdNTS}(\alpha, \theta; B)$. Then we obtain the ARMA-GARCH-NTS model. Suppose that the parameter α and θ are fixed real numbers, and parameter B is replaced to a random variable, then we obtain a new time series model. In this paper, we assume that

- $(\epsilon_t)_{t=1,2,\cdots}$ is not i.i.d, but $\epsilon_{t|t-1} \sim \text{stdNTS}(\alpha, \theta; B_t)$,
- and $(B_t)_{t=1,2,\cdots}$ is given by a ARIMA(1,1,0) model as follows:

$$B_{t+1} = B_t + \Delta B_{t+1}$$
$$\Delta B_{t+1} = a_0 + a_1\Delta B_t + \sigma_Z Z_{t+1},$$

where $a_0, a_1 \in \mathbb{R}$, $|a_1| < 1$, $\sigma_Z > 0$, and $(Z_t)_{t=1,2,\cdots}$ is i.i.d with $Z_t \sim N(0, 1)$.

This time series model is referred to as the *Stochastic Tails ARMA-GARCH-NTS* model or shortly the *StoT-NTS* model.

Note that, the conditional skewness of $\sigma_t\epsilon_t$ is given as

$$S(\sigma_t\epsilon_t|\mathcal{F}_{t-1}) = S(\epsilon_t|\mathcal{F}_{t-1})$$
$$= \sqrt{\frac{2-\alpha}{2\theta}} B_t \left(3(1 - B_t^2) + \frac{4-\alpha}{2-\alpha} B_t^2\right)$$

by (3). Moreover, the conditional variance of variance for $\sigma_{t+1}\epsilon_{t+1}$ is

$$\begin{aligned}\mathrm{var}(\mathrm{var}(\sigma_{t+1}\epsilon_{t+1}|\mathcal{F}_t)|\mathcal{F}_{t-1}) &= \xi^2(\sigma_{t|\mathcal{F}_{t-1}})^4 E[\epsilon^4_{t|\mathcal{F}_{t-1}}]\\ &= \xi^2(\kappa+\xi\sigma^2_{t-1}\epsilon^2_{t-1}+\zeta\sigma^2_{t-1})^2 K(\epsilon_t|\mathcal{F}_{t-1}).\end{aligned}$$

Since $\epsilon_{t|t-1} \sim \mathrm{stdNTS}(\alpha,\theta;B_t)$, we obtain

$$\begin{aligned}&\mathrm{var}(\mathrm{var}(\sigma_{t+1}\epsilon_{t+1}|\mathcal{F}_t)|\mathcal{F}_{t-1})\\ &= \frac{(2-\alpha)}{2\theta}\xi^2(\kappa+\xi\sigma^2_{t-1}\epsilon^2_{t-1}+\zeta\sigma^2_{t-1})^2\\ &\times\left((\alpha-4)(\alpha-6)\left(\frac{B_t^2}{2-\alpha}\right)^2+\left((24-6\alpha)\left(\frac{B_t^2}{2-\alpha}\right)+3(1-B_t^2)\right)(1-B_t^2)\right),\end{aligned}$$

by (4). Hence, the StoT-NTS process captures the time varying skewness and time varying vol-of-vol for the random variable B_t.

3.1. *ARMA-GARCH parameter estimation*

We estimate model parameters using S&P 500 index daily log-return data. ARMA(1,1)-GARCH(1,1) parameters are estimated for every 3,607 working days between December 26, 2003 to June 1, 2018. In each estimation, we use 1,000 historical log-returns by the current day. For example,

- at December 26, 2003, we estimate those parameters using 1,000 daily log-returns from January 4, 2000 to December 26, 2003,
- at June 1, 2018, we estimate those parameters using 1,000 daily log returns from May 7, 2014 to June 1, 2018.

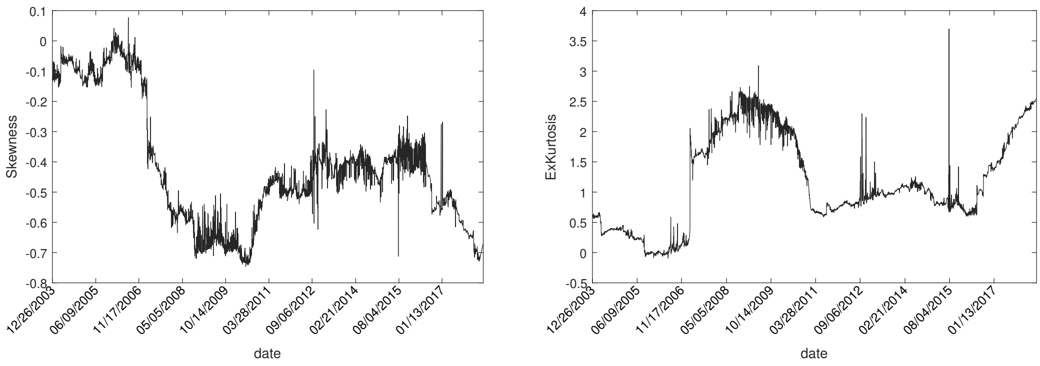
Then we obtain 3,607 residual sets. Each residual set contains 1,000 elements extracted from the estimation. Let $R_1, R_2, \cdots, R_{3607}$ be those residual sets. For instance, R_1 is the residual set extracted from the ARMA(1,1)-GARCH(1,1) estimation at December 26, 2003, and R_{3607} is the residual set extracted from the estimation at June 1, 2018. We calculate empirical skewness $\mathrm{S}(R_t)$ and empirical excess kurtosis $\mathrm{K}(R_t)$ for $R_t \in \{R_1,\ R_2,\ \cdots,\ R_{3706}\}$. Then we obtain the skewness time series $(\mathrm{S}(R_t))_{t=1,2,\cdots,3607}$ and excess kurtosis time series $(\mathrm{K}(R_t))_{t=1,2,\cdots,3607}$, which are presented in Figure 2(a) and Figure 2(b), respectively. Moreover, we plot pairs of excess kurtosis and skewness $(S(R_t), K(R_t))$ for $t \in \{1,2,\cdots,3607\}$ as Figure 3. We found that, negative skewness leads large excess kurtosis, and small excess kurtosis follows zero skewness.

3.2. *Fit parameters α and θ*

We fit α and θ of the stdNTS process as follows:

- Select one $\alpha \in (0,2)$ and one $\theta > 0$. Let $M_{\alpha,\theta} = \{(\mathrm{S}(\epsilon),\mathrm{K}(\epsilon))\,|\,\epsilon \sim \mathrm{stdNTS}(\alpha,\theta;B)$ for $B \in [-1,1]\}$.
- Applying interpolation for $M_{\alpha,\theta}$, we define a function $f_{\alpha,\theta}$ from skewness to excess kurtosis. That is,

$$f_{\alpha,\theta}(\mathrm{S}(\epsilon)) = \mathrm{K}(\epsilon) \text{ for } (\mathrm{S}(\epsilon),\mathrm{K}(\epsilon)) \in M_{\alpha,\theta}.$$



(a) Skewness (b) Excess Kurtosis

Figure 2. Time series of empirical skewness (a) and excess kurtosis (b) for each residual sets $R_1, R_2, \cdots, R_{3607}$.

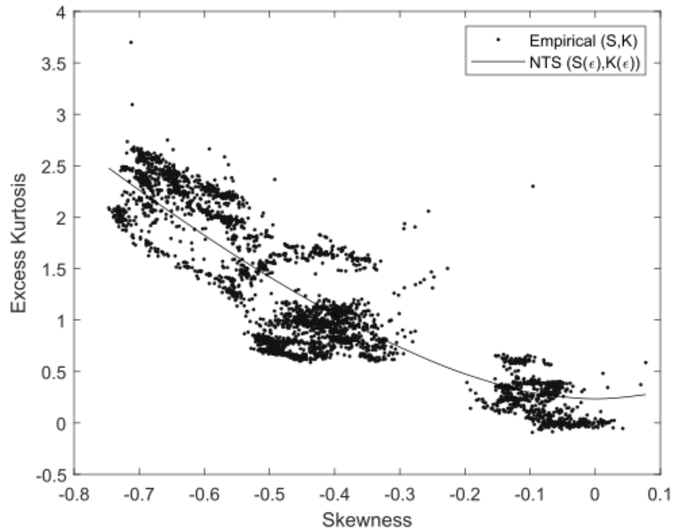


Figure 3. Dots are empirical excess kurtosis values and their corresponding empirical skewness values for set of residuals $R_t \in \{R_1, R_2, \cdots, R_{3607}\}$. The solid curve is the curve of excess kurtosis and skewness for $\epsilon \sim \text{stdNTS}(\alpha, \theta; B)$ with estimated parameters $\alpha = 1.8043$ and $\theta = 1.2544$.

- Find optimal (α^*, θ^*) minimize the square error for the empirical data as

$$(\alpha^*, \theta^*) = \arg\min_{(\alpha,\theta)} \sum_{t=1}^{T} \left[f_{\alpha,\theta}(\mathrm{S}(R_t)) - \mathrm{K}(R_t)\right]^2 / T$$

  where $T = 3607$.

By the fitting method, we obtained $(\alpha^*, \theta^*) = (1.8043, 1.2544)$, and the solid curve in Figure 3 is the function f_{α^*,θ^*}.

3.3. *Fit parameters for the time series* $(B_t)_{t\geq 0}$

We fix parameters $\alpha = 1.8043$ and $\theta = 1.2544$, and fit parameter B of stdNTS to the daily residual set R_t for $t \in \{1, 2, \cdots, 3607\}$. In this parameter fit, we find the empirical cdf F_t^{emp} using

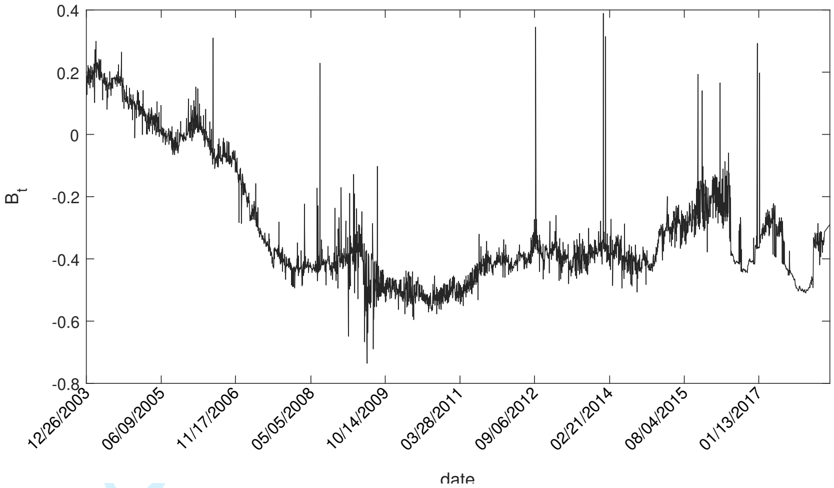


Figure 4. The time series of the estimated B_t for each residual set in $\{R_1, R_2, \cdots, R_{3607}\}$

Table 1. AR(1) Parameter Estimation

(a)

	Value	Standard Error	t-statistic	p-value
c_B	-0.00018989	0.00095468	-0.1989	0.84234
a_B	-0.47936	0.003392	-141.32	0
σ_B^2	0.0028331	$1.4057 \cdot 10^{-5}$	201.55	0

(b)

	Value	Standard Error	t-statistic	p-value
a_B	-0.47935	0.0033899	-141.4	0
σ_B^2	0.0028331	$1.3057 \cdot 10^{-5}$	216.99	0

KS-density for R_t and find B using the least square curve fit as

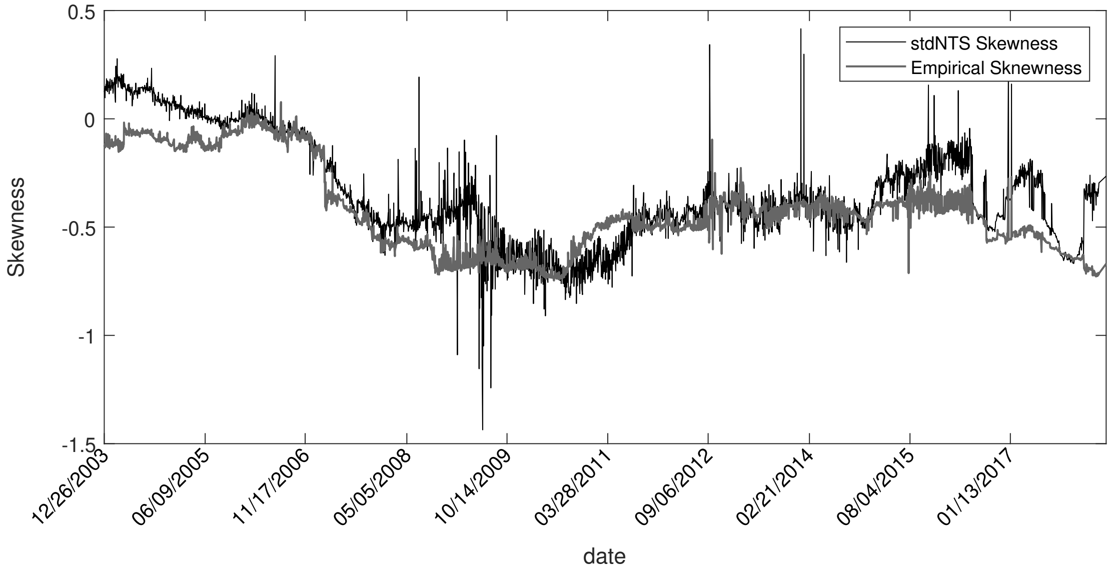
$$B_t = \arg\min_B \sum_{x_k \in R_t} (F(x_k; \alpha, \theta, B) - F_t^{emp}(x_k))^2$$

where $F(x; \alpha, \theta, B)$ is the CDF of stdNTS$(\alpha, \theta; B)$. Figure 4 presents the time series of the estimated B_t for daily residual R_t with $t \in \{1, 2, \cdots, 3607\}$. Figure 5(a) exhibits the empirical skewness time series and the skewness time series of stdNTS$(\alpha,\ \theta;\ B_t)$, and Figure 5(b) provides the empirical excess kurtosis time series and the excess kurtosis of stdNTS$(\alpha,\ \theta;\ B_t)$, where $\alpha = 1.8043$, $\theta = 1.2544$ and B_t in Figure 4.

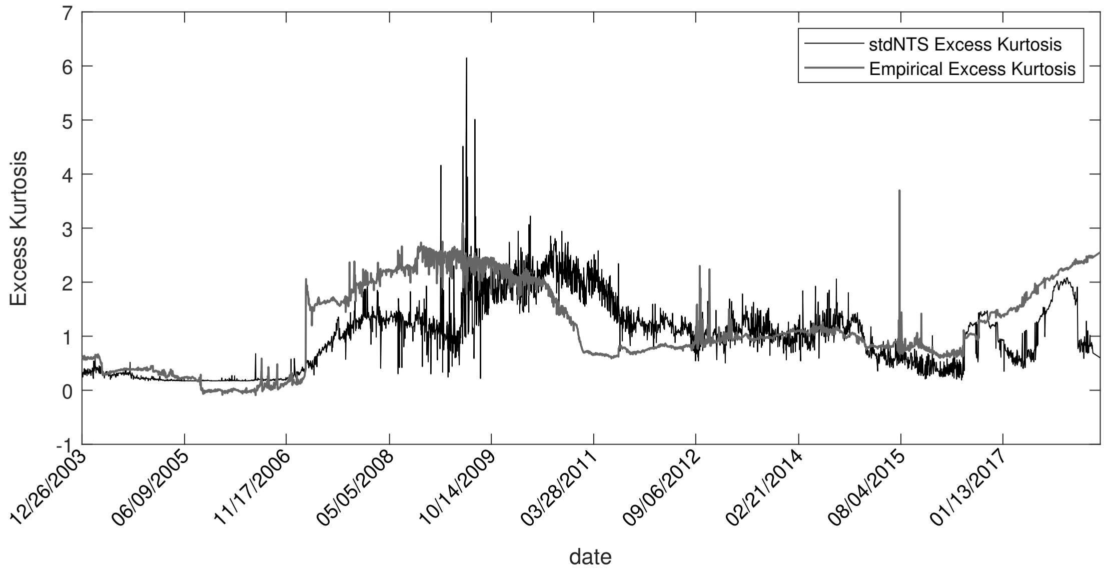
We apply the ARIMA(1,1,0) model to the time series $(B_t)_{t\geq 0}$ given in Figure 4 as

$$\Delta B_{t+1} = c_B + a_B \Delta B_t + \sigma_B Z,$$

where $\Delta B_{t+1} = B_{t+1} - B_t$. We obtain the ARIMA(1,1,0) parameters as (a) of Table 1. The constant c_B is not significant at 5% significant level, and hence we can set $c_B = 0$. The AR parameter a_B and the variance σ_B^2 are significant. Set the constant $c_B = 0$, and re-estimate ARIMA(1,1,0) we obtain (b) of Table 1 which is similar to (a). We observe the negative AR parameter, that is, ΔB_t is mean reverting.



(a) Skewness



(b) Excess Kurtosis

Figure 5. Gray curses are time series of empirical skewness and excess kurtosis and black curses are time series of stdNTS skewness (a) and excess kurtosis (b) for each residual set in $\{R_1, R_2, \cdots, R_{3607}\}$.

4. Option Pricing on the StoT-NTS Model

Let $(S_t)_{t\in\{0,1,\cdots,T^*\}}$ be the underlying asset price process and $(y_t)_{t\in\{0,1,2,\cdots T^*\}}$ be the underlying asset log return process ($y_0 = 0$) with $y_t = \log(S_t/S_{t-1})$ where $T^* < \infty$ in the time horizon. Under the physical measure $\mathbb{P} = \bigoplus_{t=1}^{T^*} \mathcal{P}_t$, $(y_t)_{t\in\{0,1,2,\cdots T^*\}}$ is supposed to follow the StoT-NTS model:

$$\begin{cases} y_{t+1} = \mu_{t+1} + \sigma_{t+1}\epsilon_{t+1|t} \\ \mu_{t+1} = c + ay_t + b\sigma_t\epsilon_{t|t-1} \\ \sigma_{t+1}^2 = \kappa + \xi\sigma_t^2\epsilon_{t|t-1}^2 + \zeta\sigma_t^2 \end{cases}$$

where $\epsilon_{t+1|t} \sim \text{stdNTS}(\alpha, \theta; B_{t+1})$, with

$$B_{t+1} = B_t + a_B \Delta B_t + \sigma_B Z_{t+1}, \quad Z_{t+1} \sim N(0,1)$$

for $t \in \{0, 1, 2, \cdots T^*\}$. Here, $\mathcal{T}$, W and $(Z_t)_{t\in\{1,2,\cdots,T^*\}}$ are mutually independent, and ϵ_0 and ΔB_0 are real constants. Let $(r_t)_{t\in\{1,2,\cdots,T^*\}}$ be sequence of the daily risk-free rate of return. There is risk-neutral measure $\mathbb{Q} = \bigoplus_{t=1}^{T^*} \mathcal{Q}_t$ such that

- $\eta_{t+1|t} = \lambda_{t+1} + \epsilon_{t+t|t}$
  where $\lambda_{t+1} = \frac{\mu_{t+1} - r_{t+1} + w_{t+1}}{\sigma_{t+1}}$ with $\omega_{t+1} = \log\left(\phi_{stdNTS(\alpha,\theta,B_t)}(-i\sigma_{t+1})\right)$
- $\eta_{t+1|t} \sim \text{stdNTS}(\alpha, \theta; B_t)$ under the measure $\mathbb{Q}$ with

$$B_{t+1} = B_t + a_B \Delta B_t + \sigma_B Z_{t+1}, \quad Z_{t+1} \sim N(0,1), t = 0, 1, \cdots, T^*.$$

Hence we have

$$\begin{cases} y_{t+1} = r_{t+1} - \omega_{t+1} + \sigma_{t+1}\eta_{t+1|t} \\ \sigma_{t+1}^2 = \kappa + \xi\sigma_t^2(\eta_{t|t-1} - \lambda_t)^2 + \zeta\sigma_t^2 \end{cases}$$

which is the risk-neutral price process.

Under the risk-neutral measure $\mathbb{Q}$, the underlying asset price is $S_t = S_0 e^{\sum_{j=0}^{t} y_j}$ for $t \in \{0, 1, 2, \cdots, T^*\}$. The European option with a payoff function $H(S(T))$ at the maturity T with $t \le T \le T^*$ is given by

$$E_{\mathbb{Q}}\left[e^{-r(T-t)}H(S(T))|\mathcal{F}_t\right] = E_{\mathbb{Q}}\left[e^{-r(T-t)}H(S_t e^{\sum_{j=t}^{T} y_j})|\mathcal{F}_t\right].$$

For example, European vanilla call and put price with strike price K and time to maturity T at time $t = 0$ are

$$C(K,T) = E_{\mathbb{Q}}\left[e^{-rT}\max\{S_0 e^{\sum_{j=0}^{T} y_j} - K, 0\}\right]$$

and

$$P(K,T) = E_{\mathbb{Q}}\left[e^{-rT}\max\{K - S_0 e^{\sum_{j=0}^{T} y_j}, 0\}\right]$$

respectively.

4.1. *Monte-Carlo Simulation and Calibration*

Assume $r_t = r$ and $\lambda_t = \lambda$ constant, to simplify the model. Let M be the number of scenarios and T be the time to maturity as a positive integer value, say days to maturity.

- Step 1
Generate a set of uniform random numbers between 0 and 1 ($\mathcal{U}(0,1)$), and two sets of independent standard normal ($N(0,1)$) random numbers

$$u_{m,n} \sim \mathcal{U}(0,1), x_{m,n} \sim N(0,1) \text{ and } z_{m,n} \sim N(0,1)$$

for $m = \{1, 2, \cdots, M\}$ and $n = \{1, 2, \cdots, T\}$.
- Step 2
Generate the tempered stable subordinator by inverse transform algorithm, as

$$\tau_{m,n} = F^{inv}_{TS(\alpha,\theta)}(u_{m,n})$$

where $F^{inv}_{TS(\alpha,\theta)}$ is the inverse CDF of tempered stable subordinator with parameter (α, θ).
- Step 3
Simulate $(B_t)_{0\le t\le T}$ as $(B_{m,n})_{m\in\{1,2,\cdots,M\},n\in\{1,2,\cdots,T\}}$, where

$$B_{m,n} = B_{m,n-1} + a_B(B_{m,n-1} - B_{m,n-2}) + \sigma_B z_{m,n}$$

and $B_{m,0}$ is B_0 value at current time, and $B_{m,1} - B_{m,0} = 0$.
- Step 4
Using (2), we simulate random number $(\eta_t)_{0\le t\le T}$ as $(\eta_{m,n})_{m\in\{1,2,\cdots,M\},n\in\{1,2,\cdots,T\}}$, where

$$\eta_{m,n} = B_{m,n}\sqrt{\frac{2\theta}{2-\alpha}}(\tau_{m,n} - 1) + x_{m,n}\sqrt{(1 - B^2_{m,n})\tau_{m,n}}.$$

- Step 5
Generate σ_t,

$$\sigma_{m,n} = \sqrt{\kappa + \xi\sigma^2_{m,n-1}(\eta_{m,n-1} - \lambda)^2 + \zeta\sigma^2_{m,n-1}}$$

$\sigma_{m,0}$ is the currently observed volatility, and generate y_t using σ_t by GARCH option pricing model as follows

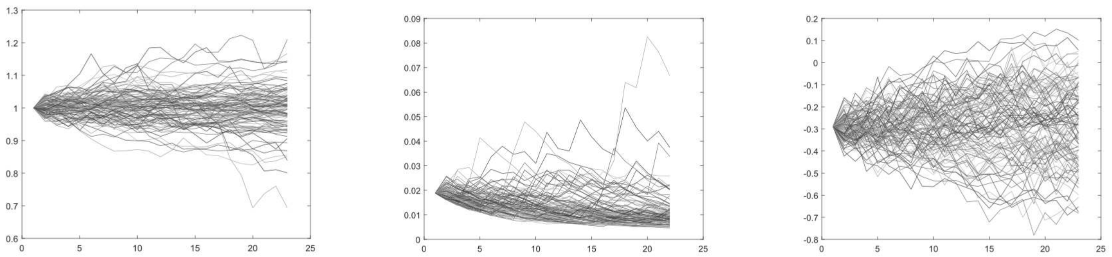
$$y_{m,n} = r - w_{m,n} + \sigma_{m,n}\eta_{m,n},$$

where $\omega_{m,n} = \log\left(\phi_{stdNTS(\alpha,\theta,B_{m,n})}(-i\sigma_{m,n})\right)$
- Step 6
The price process is obtained by

$$S_{m,n} = S_0 \exp\left(\sum_{j=1}^{n} y_{m,j}\right),$$

for $m = \{1, 2, \cdots, M\}$ and $n = \{1, 2, \cdots, T\}$.

For example, let GARCH parameters be $\kappa = 4.4115 \cdot 10^{-6}$, $\xi = 0.2289$, $\zeta = 0.7177$, ARIMA(1,1,0) parameters for (B_t) be $a_B = -0.4793$, $\sigma_B = 0.0532$ and $B_0 = -0.2895$, and tempered stable subordinator parameters be $\alpha = 1.8245$ and $\theta = 1.5063$. Set initial values of return, residual and volatility as $y_0 = 0.0373$, $\epsilon_0 = 3.6851$ and $\sigma_0 = 0.0096$, respectively. Assume that $r = (1/250)\%$, $d = 0$, and $\lambda = 0$, and generate the sample path using the algorithm for $T = 22$ and $M = 100$. Then we obtain the sample path of $(S_{m,n})$ for $S_0 = 1$, $(\sigma_{m,n})$, and $(B_{m,n})$ as Figure 6(a), Figure 6(b), and Figure 6(c), respectively.



(a) Simulated $(S_{m,n})$ (b) Simulated $(\sigma_{m,n})$ (c) Simulated $(B_{m,n})$

Figure 6. Simulated $(S_{m,n})$, $(\sigma_{m,n})$, and $(B_{m,n})$.

The European call option and put option prices with the strike price K and time to maturity T can calculated by the simulated price process as follows:

$$C(K,T) = \frac{e^{-rT}}{M}\sum_{m=1}^{M}\max\{S_{m,T}-K,0\}$$

$$P(K,T) = \frac{e^{-rT}}{M}\sum_{m=1}^{M}\max\{K-S_{m,T},0\}.$$

4.2. *Calibration*

We calibrate the StoT-NTS parameters using the S&P 500 index call and put data for the second Wednesday of each month from January 2016 to December 2017. For each calibration date, we use the GARCH parameters estimated in Section 3.1. Table 2 provides those GARCH parameters, and volatility (σ_0) and and residual (ϵ_0) observed of each date. Daily risk free rates of return and daily continuous dividend rates are also presented in base-point (bp) unit. We calculate other parameters (α, θ, a_B, σ_B, B_0, and λ) for call and put out-of-the-money (OTM) option prices. We generate one set of uniform random numbers and two sets of standard normal random numbers in Step 1 for $M = 10,000$ and $T = 90$, and fix them. After then find parameters such that minimize the mean square errors between the model price and the market prices:

$$\min_{\Theta}\left(\sum_{K_n<S_0,\ T_n<90}(P(K_n,T_n)-P_{market}(K_n,T_n))^2+\sum_{K_n>S_0,\ T_n<90}(C(K_n,T_n)-C_{market}(K_n,T_n))^2\right),$$

for $\Theta = (\alpha,\ \theta,\ a_B,\ \sigma_B,\ B_0,\ \lambda)$, where S_0 is S&P 500 index price of the given Wednesday and $P_{market}(K_n,T_n)$ and $C_{market}(K_n,T_n)$ are mid-price of observed bid and ask prices for the call and put on the given day with strike price K_n and time to maturity T_n.

For example, Figure 7 exhibits the market prices and calibrated the StoT-NTS model prices for the OTM call and puts on 5/10/2017. The calibrated GARCH-NTS model prices obtained by the simulation method are presented in the figure as a benchmark model. The GARCH-NTS model is option pricing model as

$$\begin{cases} y_{t+1} = r_{t+1} - \omega_{t+1} + \sigma_{t+1}\eta_{t+1|t} \\ \sigma_{t+1}^2 = \kappa + \xi\sigma_t^2(\eta_{t|t-1}-\lambda_t)^2 + \zeta\sigma_t^2 \end{cases},$$

where $\eta_{t+1|t} \sim \text{stdNTS}(\alpha,\theta,B)$ with constant $B \in \mathbb{R}$ (See Kim *et al.* (2010) and Rachev *et al.* (2011) for more details). Daily risk free rate of return and daily dividend rate of S&P 500 index of the day are $r = 0.3841$bp and $d = 0.7148$bp, respectively. GARCH parameters estimated historical S&P 500 index return by 5/10/2017 are $(\zeta, \xi, \kappa) = (0.7237, 0.1979, 4.9418 \cdot 10^{-6})$. The volatility and residual of the day is $\sigma_0 = 0.0046$ and $\epsilon_0 = 0.1651$, respectively. Calibrated standard NTS parameters of GARCH-NTS model are $(\alpha, \theta, B) = (0.4936, 0.1077, -0.5926)$ and $\lambda = 0.5026$, while calibrated parameters of the StoT-NTS model are $(\alpha,\ \theta,\ a_B,\ \sigma_B,\ B_0) = (0.4638,\ 0.1109,\ 0.2513,\ 0.0317,\ -0.6584)$ and $\lambda = 0.5304$. Other calibrated parameters for the second Wednesday of each month from January 2016 to December 2017 are presented in Table 3.

For the performance analysis, we use four error estimators, the average absolute error (AAE), the average absolute error as a percentage of the mean price (APE), the average relative percentage error (ARPE), and the square root of mean square relative error (RMSRE), defined as follows,

$$\text{AAE} = \sum_{n=1}^{N} \frac{|P_n - \widehat{P}_n|}{N}, \quad \text{APE} = \frac{\text{AAE}}{\sum_{n=1}^{N} \frac{P_n}{N}},$$

$$\text{ARPE} = \sum_{n=1}^{N} \frac{|P_n - \widehat{P}_n|}{N P_n}, \quad \text{RMSRE} = \sqrt{\sum_{n=1}^{N} \frac{(P_n - \widehat{P}_n)^2}{N P_n^2}},$$

where $\widehat{P}_n$ and P_n are model prices and observed market prices of options (OTM calls or OTM puts) with strikes K_n, time to maturity T_n, $n \in \{1, \ldots, N\}$, and N is the number of observed prices. Those four error estimators of the GARCH-NTS and the StoT-NTS models are presented in Table 4. According to the table, we can see that all the error estimators for the StoT-NTS model are less than corresponding error estimators of the GARCH-NTS model except the cases of 01/13/2016 and 4/12/2017, on which two model errors are similar.

To verify that the StoT-NTS model performs better than the benchmark GARCH-NTS model, we perform the simple hypothesis tests for APE, and ARPE. Since AAE and APE have the same t-statistic values, we do not need to test both, but we present only APE case test. RMSRE is also omitted in this hypothesis test, since it is not linear. Instead of the hypothesis "the StoT-NTS model performs better than the benchmark GARCH-NTS model", we use the following equivalent hypothesis:

$$H_0 : \mu_{NTS} \le \mu_{StoT} \quad \text{vs} \quad H_1 : \mu_{NTS} > \mu_{StoT}$$

or

$$H_0 : \mu_{NTS} - \mu_{StoT} \le 0 \quad \text{vs} \quad H_1 : \mu_{NTS} - \mu_{StoT} > 0$$

where μ_{StoT} and μ_{NTS} are means of calibration errors for the StoT-NTS model and GARCH-NTS model, respectively.

Let N be the number of observed prices. Let P_n be observed market prices of option and $\widehat{P}_n^{StoT}$ and $\widehat{P}_n^{NTS}$ be model prices for the StoT-NTS model and the GARCH-NTS model, respectively, for $n \in \{1, \ldots, N\}$. Then calibration errors are defined by

$$e_{APE}^{StoT}(n) = \frac{|P_n - \widehat{P}_n^{StoT}|}{\sum_{n=1}^{N} \frac{P_n}{N}} \quad \text{and} \quad e_{AAE}^{NTS}(n) = \frac{|P_n - \widehat{P}_n^{NTS}|}{\sum_{n=1}^{N} \frac{P_n}{N}}$$

$$e_{ARPE}^{StoT}(n) = \frac{|P_n - \widehat{P}_n^{StoT}|}{P_n} \quad \text{and} \quad e_{AAE}^{NTS}(n) = \frac{|P_n - \widehat{P}_n^{NTS}|}{P_n}$$

for APE, and ARPE, respectively. We perform the t-test for the hypothesis test for the following two cases:

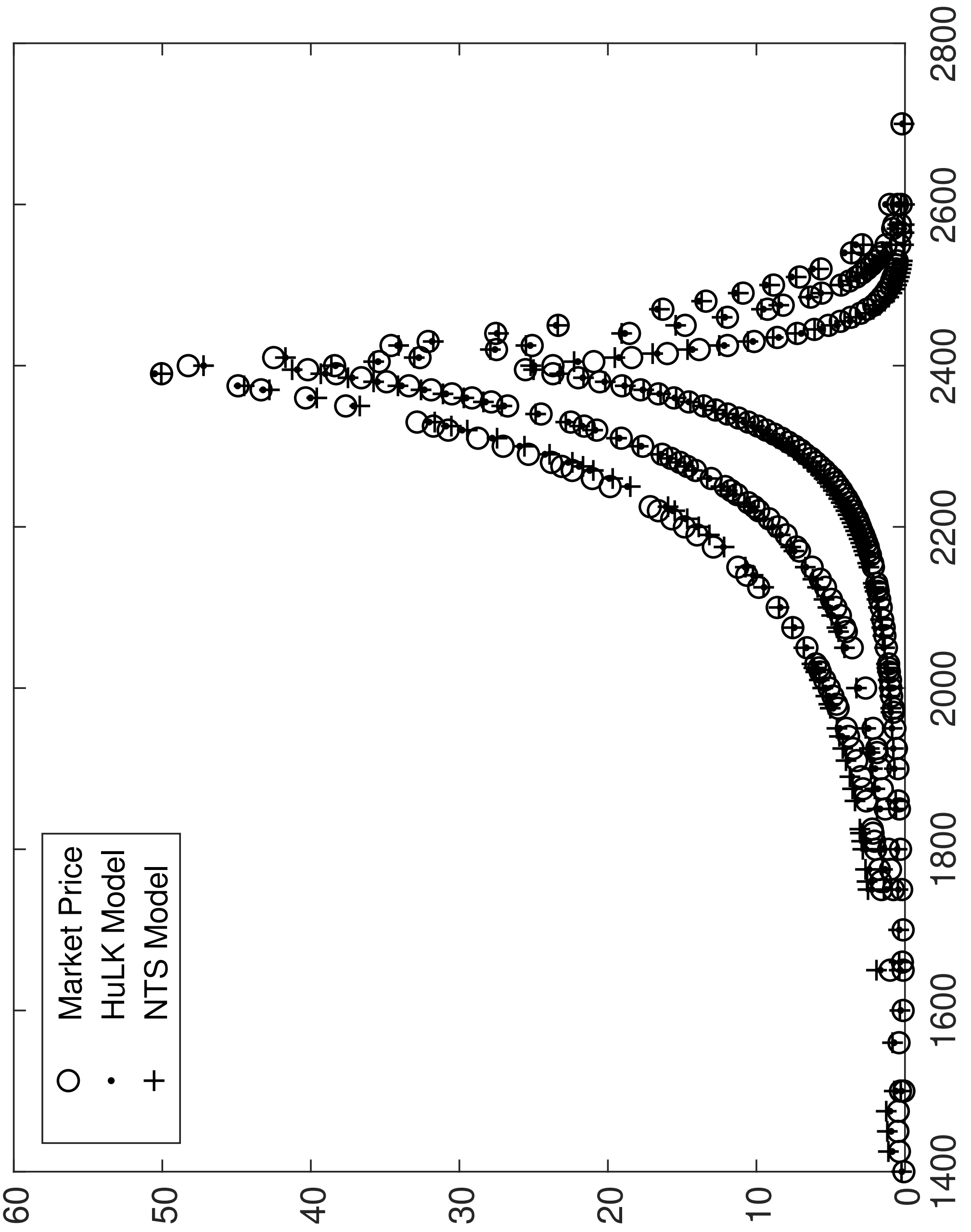


Figure 7. OTM call and put option prices and calibration result for 5/10/2017. Circles (○) are market prices of calls and puts. Dot (·) and plus (+) marks are calibrated the StoT-NTS model and the GARCH-NTS model prices, respectively.

- For APE, we set $\mu_{StoT} = E[e^{StoT}_{APE}]$ and $\mu_{NTS} = E[e^{NTS}_{APE}]$. Note that μ_{StoT} and μ_{NTS} are APE's of the StoT-NTS model and the GARCH-NTS model, respectively.
- For ARPE, we set $\mu_{StoT} = E[e^{StoT}_{ARPE}]$ and $\mu_{NTS} = E[e^{NTS}_{ARPE}]$. Note that μ_{StoT} and μ_{NTS} are ARPE's of the StoT-NTS model and the GARCH-NTS model, respectively.

The t-statistic and corresponding p-values of the tests in Table 5. Except the date 01/12/2016, 01/11/2017, and 04/12/2017, H_0 is rejected for the APE case. Except the date 01/12/2016, H_0 is rejected for the ARPE case. In the bottom line of the table, we present the result of those hypothesis tests for all market option prices and model prices we observed in this investigation. Considering total samples of every date in this investigation, H_0 is rejected for APE and ARPE. We can conclude that the StoT-NTS model calibration performs typically better than the GARCH-NTS calibration in this investigation.

5. Conclusion

In this paper, we present the StoT-NTS model obtained by taking a stochastic process for the parameter B in NTS process. The model has the stochastic exponential tails, and it deduces the stochastic skewness and stochastic kurtosis of the residual of ARMA-GARCH-NTS model, and hence it captures the time-varying vol-of-vol of the stock or index return time series. Through the empirical test of S&P 500 index return data, we observe that the skewness of the residual is typically negative. Also, if the skewness is decreasing, then the excess kurtosis of residual is increasing. The NTS distribution can describe this phenomenon by controlling the shape parameter B. By applying the ARIMA(1,1,0) model for the parameter B for time t, the StoT-NTS model describes the stochastic skewness and stochastic kurtosis empirically observed in S&P 500 index return data. The StoT-NTS option pricing model is also discussed as an application of the StoT-NTS model. We present the Monte-Carlo simulation technique based on the model and calibrate the model to the S&P 500 option prices observed in the market. In this empirical investigation, the StoT-NTS option pricing model performs mostly better than the benchmark GARCH-NTS option pricing model, since the former captures the time-varying vol-of-vol in the risk-neutral market, but the latter does not.

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Table 2. GARCH Parameters, daily risk-free rate of return, and daily dividend rate

Date	ζ	ξ	κ	σ_0	ϵ_0	r (bp)	d (bp)
01/13/2016	0.7328	0.1641	$7.1125 \cdot 10^{-6}$	0.0108	−2.6226	0.1564	0.8696
02/10/2016	0.7396	0.1661	$6.7564 \cdot 10^{-6}$	0.0119	−0.2969	0.1596	0.7830
03/09/2016	0.7502	0.1626	$6.3634 \cdot 10^{-6}$	0.0088	0.5246	0.1601	0.9031
04/13/2016	0.7529	0.1601	$6.2540 \cdot 10^{-6}$	0.0079	1.2029	0.1611	0.9105
05/11/2016	0.7410	0.1679	$6.4898 \cdot 10^{-6}$	0.0076	−1.3254	0.1636	0.8990
06/08/2016	0.7308	0.1727	$6.5207 \cdot 10^{-6}$	0.0056	0.4959	0.1653	0.9030
07/06/2016	0.6723	0.2044	$8.5048 \cdot 10^{-6}$	0.0109	0.4533	0.1778	0.9033
08/10/2016	0.6882	0.1981	$7.7233 \cdot 10^{-6}$	0.0058	−0.5718	0.1800	0.8830
09/07/2016	0.6723	0.2308	$7.1132 \cdot 10^{-6}$	0.0052	−0.2410	0.1800	0.8500
10/12/2016	0.6537	0.2152	$8.9111 \cdot 10^{-6}$	0.0083	0.0683	0.1852	0.8596
11/09/2016	0.6960	0.2035	$6.8241 \cdot 10^{-6}$	0.0099	1.0756	0.1855	0.8332
12/07/2016	0.6957	0.2036	$6.8541 \cdot 10^{-6}$	0.0056	2.2585	0.2002	0.7851
01/11/2017	0.6783	0.1998	$8.0278 \cdot 10^{-6}$	0.0057	0.3904	0.2909	0.7621
02/08/2017	0.6961	0.1942	$7.1116 \cdot 10^{-6}$	0.0055	0.0644	0.2919	0.8149
03/08/2017	0.6854	0.2051	$7.1641 \cdot 10^{-6}$	0.0062	−0.4119	0.2981	0.7584
04/12/2017	0.7035	0.1936	$6.4914 \cdot 10^{-6}$	0.0050	−0.9048	0.3841	0.6963
05/10/2017	0.7237	0.1979	$4.9418 \cdot 10^{-6}$	0.0046	0.1651	0.3841	0.7148
06/07/2017	0.6954	0.1935	$6.7354 \cdot 10^{-6}$	0.0055	0.1430	0.3952	0.7269
07/05/2017	0.7041	0.1899	$6.3665 \cdot 10^{-6}$	0.0061	0.1588	0.4830	0.7274
08/09/2017	0.7042	0.2067	$5.3908 \cdot 10^{-6}$	0.0046	−0.1730	0.4853	0.7466
09/06/2017	0.7075	0.2055	$5.2737 \cdot 10^{-6}$	0.0062	0.4306	0.4845	0.7544
10/11/2017	0.6885	0.2128	$5.8454 \cdot 10^{-6}$	0.0048	0.3022	0.4883	0.7023
11/08/2017	0.6990	0.2104	$5.3214 \cdot 10^{-6}$	0.0045	0.2355	0.4885	0.6437
12/06/2017	0.6974	0.2113	$5.3551 \cdot 10^{-6}$	0.0055	−0.1018	0.4980	0.6068

(bp $= 10^{-4}$)

Table 3. Calibrated Parameters

Date	Model	Parameters					
		α	θ	a_B	σ_B	B or B_0	λ
01/13/2016	GARCH-NTS	0.3157	2.8308			−0.7533	0.6541
	HuLK-NTS	0.3157	2.8308	−0.9990	0.0039	−0.7543	0.6541
02/10/2016	GARCH-NTS	1.0877	17.3946			−0.5617	0.7547
	HuLK-NTS	1.0909	17.3939	0.0010	0.0533	−0.5808	0.7567
03/09/2016	GARCH-NTS	1.2678	4.5884			−0.9682	0.6730
	HuLK-NTS	1.2708	4.6373	0.2819	0.0096	−0.9898	0.6709
04/13/2016	GARCH-NTS	0.6467	0.4114			−0.8137	0.6016
	HuLK-NTS	0.7367	0.4107	0.8517	0.0020	−0.8305	0.6023
05/11/2016	GARCH-NTS	0.4776	0.8305			−0.8493	0.5565
	HuLK-NTS	0.4776	0.8305	0.2335	0.0065	−0.8675	0.5565
06/08/2016	GARCH-NTS	0.5757	0.5550			−0.9465	0.6194
	HuLK-NTS	0.5565	0.5703	0.8868	0.0028	−0.9507	0.6113
07/06/2016	GARCH-NTS	0.6141	0.4658			−0.8272	0.6163
	HuLK-NTS	0.6343	0.4300	−0.9963	0.0666	−0.8375	0.6067
08/10/2016	GARCH-NTS	0.0001	0.2165			−0.7442	0.5899
	HuLK-NTS	0.0001	0.1893	−0.0344	0.0322	−0.7478	0.6101
09/07/2016	GARCH-NTS	0.0004	0.4086			−0.8260	0.5021
	HuLK-NTS	0.0002	0.4305	0.1365	0.0152	−0.8523	0.4957
10/12/2016	GARCH-NTS	0.0002	0.9400			−0.9586	0.6031
	HuLK-NTS	0.0178	0.9569	−0.3504	0.0080	−0.9541	0.5976
11/09/2016	GARCH-NTS	0.0002	0.5519			−0.8848	0.5238
	HuLK-NTS	0.0002	0.5622	−0.0473	0.0138	−0.9000	0.5180
12/07/2016	GARCH-NTS	0.3802	0.1320			−0.4659	0.5585
	HuLK-NTS	0.5698	0.1237	−0.9961	0.0470	−0.5066	0.5708
01/11/2017	GARCH-NTS	0.0002	0.2608			−0.6486	0.4884
	HuLK-NTS	0.0002	0.2551	0.7315	0.0075	−0.6555	0.4945
02/08/2017	GARCH-NTS	0.3203	0.1201			−0.6169	0.5855
	HuLK-NTS	0.2157	0.1296	0.5971	0.0251	−0.7121	0.5985
03/08/2017	GARCH-NTS	0.9214	0.0687			−0.5017	0.5752
	HuLK-NTS	0.9214	0.0692	−0.9965	0.0268	−0.5066	0.5758
04/12/2017	GARCH-NTS	0.0001	0.5754			−0.9999	0.5935
	HuLK-NTS	0.0001	0.5936	−0.4362	0.0176	−0.9270	0.5828
05/10/2017	GARCH-NTS	0.4936	0.1077			−0.5926	0.5026
	HuLK-NTS	0.4638	0.1109	0.2513	0.0317	−0.6584	0.5304
06/07/2017	GARCH-NTS	0.4410	0.0518			−0.6267	0.6256
	HuLK-NTS	0.5993	0.0456	0.2350	0.0431	−0.7232	0.6690
07/05/2017	GARCH-NTS	0.4962	0.0571			−0.7152	0.6234
	HuLK-NTS	0.4753	0.0548	−0.3019	0.0395	−0.7358	0.6458
08/09/2017	GARCH-NTS	0.0002	0.1760			−0.8426	0.5959
	HuLK-NTS	0.0001	0.1698	−0.5737	0.0323	−0.8492	0.5968
09/06/2017	GARCH-NTS	0.0001	0.2577			−0.7896	0.5676
	HuLK-NTS	0.0001	0.2559	−0.0055	0.0280	−0.8015	0.5675
10/11/2017	GARCH-NTS	0.0001	0.0523			−0.7615	0.6511
	HuLK-NTS	0.0132	0.0502	0.1785	0.0408	−0.8713	0.6618
11/08/2017	GARCH-NTS	0.5200	0.0742			−0.6892	0.5915
	HuLK-NTS	0.5003	0.0619	−0.3800	0.0725	−0.7321	0.6341
12/06/2017	GARCH-NTS	1.1843	0.0293			−0.6654	0.7303
	HuLK-NTS	1.1843	0.0293	0.0969	0.0195	−0.6677	0.7303

Table 4. Error Estimators

Date	Model	AAE	APE	ARPE	RMSRE
01/13/2016	NTS	1.0433	0.0627	0.2086	0.3005
	HuLK	1.0439	0.0627	0.2090	0.3009
02/10/2016	NTS	2.5294	0.1303	0.3708	0.7047
	HuLK	2.4862	0.1280	0.3532	0.6844
03/09/2016	NTS	1.2637	0.0830	0.2717	0.4972
	HuLK	1.1922	0.0783	0.2535	0.4587
04/13/2016	NTS	0.7991	0.0695	0.2946	0.4321
	HuLK	0.7871	0.0685	0.2919	0.4321
05/11/2016	NTS	0.8853	0.0826	0.3687	0.6119
	HuLK	0.8786	0.0820	0.3747	0.6304
06/08/2016	NTS	0.5778	0.0560	0.2260	0.3538
	HuLK	0.5225	0.0506	0.2019	0.3162
07/06/2016	NTS	0.9900	0.0934	0.3577	0.5140
	HuLK	0.9160	0.0864	0.3331	0.4990
08/10/2016	NTS	0.8431	0.0906	0.3438	0.5258
	HuLK	0.7956	0.0855	0.3164	0.4922
09/07/2016	NTS	1.1065	0.1131	0.4689	0.7001
	HuLK	1.0700	0.1093	0.4357	0.6462
10/12/2016	NTS	1.0079	0.0725	0.3171	0.6005
	HuLK	0.9926	0.0714	0.3096	0.5873
11/09/2016	NTS	0.6946	0.0717	0.3003	0.4240
	HuLK	0.6644	0.0686	0.2909	0.4222
12/07/2016	NTS	1.0412	0.0869	0.3368	0.5429
	HuLK	0.9970	0.0832	0.3301	0.5364
01/11/2017	NTS	0.7766	0.0768	0.3218	0.4951
	HuLK	0.7651	0.0756	0.3025	0.4695
02/08/2017	NTS	0.6380	0.0640	0.2337	0.4186
	HuLK	0.6057	0.0607	0.2134	0.3566
03/08/2017	NTS	1.0550	0.0995	0.2997	0.4677
	HuLK	1.0349	0.0976	0.2794	0.4191
04/12/2017	NTS	1.0937	0.0883	0.2558	0.4066
	HuLK	1.0966	0.0885	0.2219	0.3437
05/10/2017	NTS	0.4560	0.0459	0.2194	0.4561
	HuLK	0.4002	0.0402	0.1880	0.3690
06/07/2017	NTS	0.6847	0.0664	0.2270	0.3629
	HuLK	0.5811	0.0563	0.1876	0.2907
07/05/2017	NTS	0.6090	0.0737	0.2887	0.4260
	HuLK	0.5764	0.0698	0.2540	0.3810
08/09/2017	NTS	0.5904	0.0542	0.2203	0.4140
	HuLK	0.5601	0.0515	0.1980	0.3570
09/06/2017	NTS	0.9301	0.0800	0.3284	0.5947
	HuLK	0.9006	0.0775	0.2821	0.4748
10/11/2017	NTS	0.6265	0.0701	0.2331	0.3827
	HuLK	0.5492	0.0615	0.1759	0.2905
11/08/2017	NTS	0.7517	0.0784	0.3576	0.6059
	HuLK	0.6257	0.0653	0.2810	0.4890
12/06/2017	NTS	0.8312	0.0662	0.2492	0.5251
	HuLK	0.8156	0.0650	0.2286	0.4760

Table 5. Hypothesis test for APE and ARPE

Date	N	APE			ARPE		
		$\mu_{NTS} - \mu_{HuLK}$	t-statistic	p-value	$\mu_{NTS} - \mu_{HuLK}$	t-statistic	p-value
01/13/2016	372	−0.0000	−1.2180	0.8884	−0.0004	−2.3838	0.9914
02/10/2016	308	0.0022***	3.6422	$1.35 \cdot 10^{-4}$	0.0176***	7.4625	$4.24 \cdot 10^{-14}$
03/09/2016	297	0.0047***	9.2688	0	0.0182***	5.4127	$3.10 \cdot 10^{-8}$
04/13/2016	309	0.0010***	4.3800	$5.93 \cdot 10^{-6}$	0.0027***	3.5345	$2.04 \cdot 10^{-4}$
05/11/2016	345	0.0006*	1.7110	0.0435	−0.0060	−4.0520	1.0000
06/08/2016	316	0.0054***	7.3825	$7.76 \cdot 10^{-14}$	0.0241***	8.7411	0
07/06/2016	346	0.0070***	7.3198	$1.24 \cdot 10^{-13}$	0.0246***	4.3254	$7.61 \cdot 10^{-6}$
08/10/2016	292	0.0051***	5.5679	$1.29 \cdot 10^{-8}$	0.0274***	6.4771	$4.68 \cdot 10^{-11}$
09/07/2016	300	0.0037***	5.5828	$1.18 \cdot 10^{-8}$	0.0331***	8.8502	0
10/12/2016	288	0.0011***	3.6713	$1.21 \cdot 10^{-4}$	0.0075***	3.8506	$5.89 \cdot 10^{-5}$
11/09/2016	423	0.0031***	6.9092	$2.44 \cdot 10^{-12}$	0.0094***	3.7933	$7.43 \cdot 10^{-5}$
12/07/2016	351	0.0037***	6.4276	$6.48 \cdot 10^{-11}$	0.0067*	1.6583	0.0486
01/11/2017	301	0.0011	1.5756	0.0576	0.0193***	6.9910	$1.37 \cdot 10^{-12}$
02/08/2017	203	0.0032*	2.1904	0.0142	0.0204**	2.9323	$1.68 \cdot 10^{-3}$
03/08/2017	306	0.0019***	4.3900	$5.67 \cdot 10^{-6}$	0.0203***	4.5583	$2.58 \cdot 10^{-6}$
04/12/2017	282	−0.0002	−0.1981	0.5785	0.0339***	5.5053	$1.84 \cdot 10^{-8}$
05/10/2017	259	0.0056***	5.1967	$1.01 \cdot 10^{-7}$	0.0314***	4.3879	$5.72 \cdot 10^{-6}$
06/07/2017	225	0.0101***	8.1728	$1.11 \cdot 10^{-16}$	0.0394***	5.9808	$1.11 \cdot 10^{-9}$
07/05/2017	321	0.0040***	5.1178	$1.55 \cdot 10^{-7}$	0.0348***	6.8810	$2.97 \cdot 10^{-12}$
08/09/2017	279	0.0028***	4.9184	$4.36 \cdot 10^{-7}$	0.0224***	4.9254	$4.21 \cdot 10^{-7}$
09/06/2017	272	0.0025**	2.7741	$2.77 \cdot 10^{-3}$	0.0462***	5.2929	$6.02 \cdot 10^{-8}$
10/11/2017	269	0.0087***	6.5538	$2.81 \cdot 10^{-11}$	0.0573***	7.7381	$5.00 \cdot 10^{-15}$
11/08/2017	271	0.0131***	8.8339	0	0.0765***	8.0176	$5.55 \cdot 10^{-16}$
12/06/2017	260	0.0012*	1.9172	0.0276	0.0207***	4.5108	$3.23 \cdot 10^{-6}$
total	7195	0.0034***	22.1464	0	0.0231***	23.5084	0