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On the Robust Measurement of Inequality

Xiangyu Qu *

Abstract

In practice, a dataset used for calculating inequality index does not always present in a single statistic fashion. A robust inequality measure, in the face of multi-valued statistic dataset, is needed and should take into account both inequality and imprecision concerns in a proper way. However, we find that a commonly used approach is problematic in imprecision reduction. We therefore suggest a new approach of robust inequality measure which surmounts the difficulty. This approach naturally generalizes the Atkinson and Gini indices to measure multi-valued problem. We finally axiomatize two social welfare functions which induce the robust Atkinson and the robust Gini indices.

JEL classification: D31, D63, D81

Keywords: Robust inequality index; Social welfare function; Multi-valued dataset

1 INTRODUCTION

In the wake of financial crisis, social conflicts and pandemic, inequality has been one of the major subjects in economics in recent years (Atkinson, Piketty, and Saez [2011]). While inequality is important, researchers often have to deal with a paucity of data which makes measuring it a daunting challenge. In theory, if an economic distribution within a population can be precisely and correctly summarized in a single statistic, then it is straightforward to measure inequality by, to name a few, the Atkinson [1970], the Gini [1921] and the Theil [1967] indices[1]. However, in reality, a data set we rely on typically does not present itself in a precise way such that we can directly obtain a distribution in a single statistics. For instance, individual incomes, household wealth, educational

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[1] We refer to Cowell [2011] for a survey of various inequality indices.

achievement scores and housing values are usually presented in multi-valued form to either encourage responses or protect confidentiality. In such multi-valued data, the information provided is the number of observations between lower and upper limits[2]. Therefore, measuring inequality in the face of multi-valued data is not straightforward and calls for a reliable methodology. This paper addresses the key considerations that must be made when measuring inequality of multi-valued statistic data.

The bulk of the literature on inequality assumes that if multi-valued data is present, one may reduce the multi-valued problem to the case of single-value. For example, the midpoint estimator (Heitjan [1989], Henson [1967]) has been widely used in practice. That is, replacing each individual multi-valued income by its mean or median value. Then calculating inequality index solely bases on individual mean or median values. Alternatively, it is envisaged that one may use classic inequality measurement with respect to every possible single-valued distribution to reduce the problem to a choice among a set of index values. We claim that the former approach, the midpoint reduction approach, is an unsatisfactory approach to the measurement of inequality with multiple values. We also claim that the latter approach avoid the drawback of the former one and should be a plausible approach to be applied in practice.

To see our claim, consider a society consisting of two individuals, *Ana* and *Bob*, for simplicity. Suppose that the income of Ana is investigated through survey, therefore, the only information we know is that her income is between $10,000 and $20,000. However, the income of Bob is investigated through tax account, therefore, we know exactly that his income is $15,000. According to midpoint reduction approach, income distribution between Ana and Bob is completely equalized. In other words, society ranks multi-valued distribution ([$10,000, $20,000]; $15,000) and single-valued distribution ($15,000; $15,000) indifferently. However, we argue that a reasonable social ordering should rank the latter distribution higher than the former one. Clearly, ($15,000; $15,000) is a definite equality, but not ([$10,000, $20,000]; $15,000). Since the income of Ana is not precise, society could not rule out the situation of inequality that Ana's income is not the same as Bob's. Any society who tends to avoid such imprecision should prefer the latter distribution. In this paper, we would develop a novel inequality measurement to avoid unreasonable ranking raised by middle point reduction approach.

We observe that the alternative approach may capture the preferences of aversion to both inequality and imprecision if we first calculate the inequality of possible distributions and then select one value based on the set of index values. For instance, suppose that the Gini index is the accepted

[2] Multi-valued data can be either an interval, like [$10,000, $15,000] for income bracket, or a finite possible numbers, like $\{80, 85, 90\}$ for educational scores.

measure of inequality. Suppose that lower and upper limits of Ana's income are most relevant possible incomes. Therefore, the Gini indices of lower distribution ($10,000; $15,000) and upper distribution ($20,000; $15,000) are strictly positive and larger than that of ($15,000; $15,000), which is zero. If a society adopts a (weighted) average of Gini indices of lower and upper distributions as an inequality measure between Ana and Bob, then it must rank ($15,000; $15,000) above ([$10,000, $20,000]; $15,000). We therefore aim to develop a novel measure of inequality that take into account both inequality reduction and aversion to imprecision, and in particular, that include the above mentioned method.

Our goal is to propose a class of robust measures that is a natural generalization of those widely used for the measurement of inequality with respect to single-valued distribution. More precisely, consider a multi-valued distribution F. Let $\overline{F}$ ($\underline{F}$) define the upper (lower) limit distribution. We suggest that a society could measure the inequality of F in the following way:

$$I(F) = \lambda_F \cdot \phi(\overline{F}) + (1 - \lambda_F) \cdot \phi(\underline{F}),$$

where ϕ is a measure of single-valued distribution. The parameter $0 \leq \lambda_F \leq 1$ can be interpreted as a measure of the social attitude toward inequality of upper limit distribution of F.

We start with Atkinson [1970]- Kolm [1969]- Sen [1973] approach to characterize a class of social welfare functions that would induce the above measurement. We are particularly interested in two special cases of function ϕ, namely the Atkinson index and the Gini index. On the one hand, both indices are widely used in inequality literature, the robust version of them, therefore, should be empirically relevant. On the other hand, their associated social welfare functions are theoretically meaningful. First, robust Atkinson social welfare function has a separably additive form, which is normatively appealing. In single value problem, Chambers [2012] demonstrated that if a social welfare function has a Bergenson-Samuelson form, then more inequality averse society implies more risk aversion. However, in robust form, though separably additive social welfare function is a sum of individual utilities, individual utility itself is defined on set of values rather than singleton value. Therefore, inequality aversion is related to not only risk aversion but also imprecision aversion. Second, robust Gini social welfare function is non-additive. In single-valued problem, Gini social welfare function belongs to a family of Choquet integration à la Schmeidler [1989], which is comonotonically additive. To maintain this plausible property, we extend this concept from single value to multiple values and demonstrate that our robust Gini index satisfies this property. Third, both robust indices may potentially connect to political economy models. In single-valued problem, Salas and Rodríguez [2013] showed that in a class of separably additive social welfare functions, Atkinson social welfare function accords with the majority voting scheme. In a subclass of non-additive social welfare functions, Rodríguez and Salas [2014] showed that Gini social welfare function accords with majority voting scheme. Therefore, our construction of robust social welfare function would facilitate to establish the connection between inequality and political economy models.

We furthermore seek a set of ethical axioms that characterizes the robust Atkinson and robust Gini social welfare functions. Our axioms are equally plausible in the contexts of both single-valued and multi-valued situations. That is, our axioms coincide with Atkinson axioms or Gini axioms in the face of single-valued distribution. However, our axioms would reflect both inequality and imprecision considerations in the face of multi-valued distribution. There are clearly alternative indices that can take into account both inequality and imprecision. We delay our comparison discussions about the strength and weakness with other possibilities in conclusion section.

This paper can also be viewed as an attempt to provide a complementary method for studying *epistemic* uncertainty to that developed within *subjective* uncertainty by Ben-Porath, Gilboa, and Schmeidler [1997], Gajdos and Tallon [2002], Gajdos and Maurin [2004], Chew and Sagi [2012] and many others. As Fox and Ülkümen [2011] point out, epistemic uncertainty is about the state of the world that we do not know but could know in theory, such as uncertainty due to limitations of the data; and subjective uncertainty is about the state of the world that we cannot know, such as randomness or chance. It is important to emphasize that multi-valued problem we study belongs to the category of epistemic uncertainty. Furthermore, it is not clear what the state space is under our framework. Although psychological study suggests that people intuitively distinguish between these two kinds of uncertainty, much less inequality study, in particular theoretical study, has specifically focused on epistemic uncertainty. Therefore, this paper suggests a novel inequality index with theoretical foundation under uncertainty when the state space cannot be naturally constructed.

At least since Atkinson [1970], inequality literature is well connected to decision theory. With no exception, our robust social welfare functions are related to the concept of maxmin expected utility of Gilboa and Schmeidler [1989] and other similar concepts, such as α-maxmin expected utility of Ghirardato, Maccheroni, and Marinacci [2004] and Hurwicz expected utility of Gul and Pesendorfer [2015]. However, we consider the environment where no state space is presented. In that sense, objective ambiguity model of Olszewski [2007] is closer to ours. Though concepts are similar, the motivation and application are significantly different. At a technical level, the main distinction is that we allow for non-additive measure ϕ with respect to single-valued distribution.

The next section explores a social welfare approach to construct a robust measure of inequality. We also discuss how to extend two widely used inequality indices, namely Atkinson and Gini indices, to robust indices. Section 3 focus on the robust Atkinson index and the robust Gini index. We axiomatize the robust Atkinson (Gini) social welfare function, which would induce the robust Atkinson (Gini) index. Section 4 concludes and discusses alternative measurements. The appendix contains all proofs.

2 INEQUALITY MEASUREMENT

2.1 Setup

Consider a society $\mathcal{N}$ consists of $n \geq 2$ *individuals*. Let $X = \mathbb{R}_+$ be the set of individual allocations. We denote by $\mathcal{X}$ the collection of all non-empty compact subsets of X. An allocation profile is denoted by $F = (F_1, \ldots, F_n)$, where each $F_i \in \mathcal{X}$ contains all the possible allocations of individual i. An allocation profile is *deterministic* (or called a *distribution*), written as $f = F$, if each F_i is a singleton, i.e. $F_i \in X$. Let $\mathcal{F}$ be the collection of all the possible allocation profiles and let X^n denote the set of all the deterministic allocation profiles. We denote $\mathbb{1} \in X^n$ the deterministic profile f where $f_i = 1$ for all i. If no confusion arises, we write deterministic profile $f \in F$ if $f_i \in F_i$ for each i.

For $Y, Z \in \mathcal{X}$, we write $Y \geq Z$ if $y \geq z$ for all $y \in Y$ and $z \in Z$. For $F \in \mathcal{F}$, we denote $\overline{F}$ the upper limit distribution in F if $\overline{F} \in F$ and $\overline{F}_i \geq F_i$ for all i. Similarly we denote $\underline{F}$ the lower limit distribution in F if $\underline{F} \in F$ and $\underline{F}_i \leq F_i$ for all i. Also, for $F, G \in \mathcal{F}$, we write $F \geq G$ if $F_i \geq G_i$ for all i.

For $f \in X^n$, we write $\mu(f) = \frac{1}{n}\sum_{i=1}^{n} f_i$ for the *mean* of f. Also, let $\tilde{f}$ be the deterministic allocation profile obtained from f by rearranging the allocation in an increasing order, *i.e.* $\{f_1, \ldots, f_n\} = \{\tilde{f}_1, \ldots, \tilde{f}_n\}$ and $\tilde{f}_1 \leq \ldots \leq \tilde{f}_n$.

2.2 Robust Inequality Index

To construct a robust inequality index, we adopt Atkinson [1970]- Kolm [1969]- Sen [1973] (AKS) approach that an inequality index should be a transformation of a social welfare function which emphasizes the welfare loss due to the inequality in the allocation profile. Formally, a *social welfare function* (SWF) $W : \mathcal{F} \to \mathbb{R}$ maps allocation profiles to real numbers.

In order to present a development on welfare theoretic approache to the measurement of inequality, we focus on the class of SWF which displays inequality reduction property. To this end, we assume that SWF should satisfy the following three assumptions. We say a SWF W is *Schur-*

concave on deterministic profiles if for all $f \in X^n$ and all bistochastic matrices M of order n[3], $W(fM) \geq W(f)$. We say a SWF W is *monotonic* if for all $F, G \in \mathcal{F}$, $W(F) \geq W(G)$ whenever $F \geq G$. We refer to a SWF as *regular* if it is continuous with respect to Hausdorff distance[4], monotonic and Schur-concavity on deterministic profiles. We assume throughout this section that W is regular.

Given a regular SWF W, for any allocation profile F we define the *equally distributed equivalent* $\xi(F) \in \mathbb{R}$ as follows:

$$W(\xi(F) \cdot \mathbb{1}) = W(F).$$

Therefore, $\xi(F)$ is the level of allocation which if given to each individual will make the existing profile F socially indifferent. Since W satisfies regularity conditions, this can be used to yield the equally distributed equivalent as a function $\xi : \mathcal{F} \to \mathbb{R}$. In other words, given a profile F, $\xi(F)$ can be uniquely extracted from the above equation. In particular, note ξ is also regular. Further, it is immediate to see $\xi(c \cdot \mathbb{1}) = c$ for all $c > 0$.

Due to monotonicity, for $F \in \mathcal{F}$, we have

$$\xi(\underline{F}) \leq \xi(F) \leq \xi(\overline{F}).$$

So there exists a unique $\lambda_F \in [0, 1]$ such that $\lambda_F \xi(\overline{F}) + (1 - \lambda_F)\xi(\underline{F}) = \xi(F)$. Accordingly, we propose a simple transformation of regular SWF as an index of inequality.

Definition 1. A function $I : \mathcal{F} \to \mathbb{R}$ is said to be a *robust index of inequality* if, for all $F \in \mathcal{F}$ with $\underline{F} \neq 0$,

$$I(F) = 1 - \Big\{ \lambda_F \frac{\xi(\overline{F})}{\mu(\overline{F})} + (1 - \lambda_F) \frac{\xi(\underline{F})}{\mu(\underline{F})} \Big\}. \tag{1}$$

This definition coincides with AKS whenever profile is deterministic. Note that I is defined on profiles in which each individual has zero allocation is not feasible. Our proposal is plausible because this index has important properties as the classical index requires.

[3]A $n \times n$ matrix M with nonnegative entries is called a bistochastic matrix order n if each of its rows and columns sums to unity.

[4]For every pair of deterministic allocation profiles f, g, the distance between f and g can be induced by a natural topology, written as $d(f, g)$, on $\mathbb{R}^n$. Therefore, the set of allocation profiles $\mathcal{F}$ can be equipped with Hausdorff distance in the following way: for $F, G \in \mathcal{F}$,

$$\text{dist}(F, G) = \max \big\{ \max_{f \in F} \min_{g \in G} d(f, g), \max_{g \in G} \min_{f \in F} d(f, g) \big\}.$$

Proposition 1. *A robust index of inequality I has the following properties:*

(i) *Betweeness: Each $I(F)$ lies between $I(\overline{F})$ and $I(\underline{F})$.*

(ii) *Schur convexity on deterministic profiles: $I(f) \geq I(fM)$ for every bistochastic matrix M and deterministic profile f.*

(iii) *Normalization: Each $I(F)$ lies in $[0,1]$; and $I(F) = 0$ iff $\overline{F} = c \cdot \mathbb{1}$ and $\underline{F} = c' \cdot \mathbb{1}$ for some $c \geq c' > 0$.*

We actually can rewrite index I in a weighted average of $I(\overline{F})$ and $I(\underline{F})$.

$$I(F) = \lambda_F I(\overline{F}) + (1 - \lambda_F) I(\underline{F}).$$

Using this, we can express $\xi(F)$as

$$\xi(F) = \lambda_F \big[\mu(\overline{F})(1 - I(\overline{F}))\big] + (1 - \lambda_F)\big[\mu(\underline{F})(1 - I(\underline{F}))\big].$$

As noted, the function ξ itself or any increasing transformation function of it can be regarded as a regular SWF. Thus, ξ implies and is implied inequality indices. However, $\xi(F)$ is not directly implied by $I(F)$, but through $I(\overline{F})$, $I(\underline{F})$ and $I(F)$. This welfare function is represented as an increasing function of a weighted sum of one product of the mean of upper limit distribution and the shortfall of its inequality index from unity, and another product of the mean of lower limit distribution and the shortfall of its inequality index from unity. It expresses welfare as a trade-off between equity and efficiency. Such a welfare function is referred to as a boundary reduced-form welfare function because its arguments summarize the entire distribution in terms of the mean and inequality of upper limit and lower limit distributions.

2.3 Lorenz Dominance and Robust Inequality

Lorenz [1905] uses a Lorenz curve to present deterministic allocation profile in an illuminating fashion. The Lorenz domination criterion is widely acknowledged as a fundamental principle to rank alternative profiles in terms of comparative inequality. In this subsection, we explore the extension of Lorenz domination from deterministic profiles to general profiles and develop its relation with SWF.

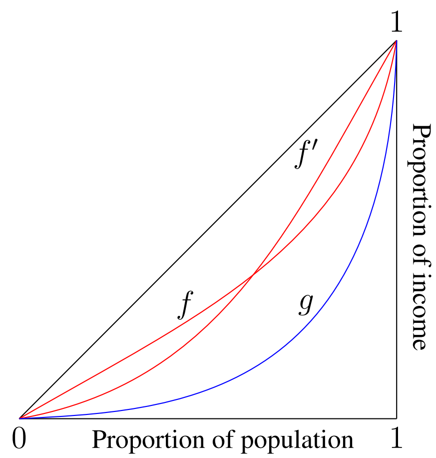

Figure 1: $\{f, f'\}$ Lorenz dominates g.

Recall that a deterministic profile f is said to *Lorenz dominate*[5] g if

$$\frac{1}{n\mu(f)} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{1}{n\mu(g)} \sum_{i=1}^{k} \tilde{g}_i,$$

for all $k = 1, 2 \ldots, n$. That is, f Lorenz dominates g if the Lorenz curve of f is nowhere below the Lorenz curve of g. Now we extend this definition on deterministic profiles to the general profiles.

Definition 2. A profile F *Lorenz dominates* another profile G, write as $F \succsim_L G$, if for every $f \in F$ and $g \in G$, f Lorenz dominates g.

A profile F Lorenz dominates G if every feasible deterministic profile in F Lorenz dominates every deterministic allocation in G. Thus, as we can see in Figure (1), if $F = \{f, f'\}$ and $G = \{g\}$, then F Lorenz dominates G. However, the ranking of profiles generated by the Lorenz domination comparison is incomplete since, assuming $F' = \{f, g\}$ and $G' = \{f', g\}$, we cannot rank F' and G' by the Lorenz domination criterion. Though, $\succsim_L$ is incomplete, but it satisfies transitivity. Below we state the relation between the Lorenz domination criterion and social welfare functions.

Proposition 2. *Let F and G be two profiles such that $\min_{f \in F} \mu(f) \geq \max_{g \in G} \mu(g)$. Then $F \succsim_L G$ if and only if $W(F) \geq W(G)$, and $W(f) \geq W(g)$ for each $f \in F$ and $g \in G$.*

This result says that a profile and any deterministic profiles in this one rank higher than another profile and any deterministic profiles in it, respectively, by regular SWF if and only if the Lorenz

[5]The classic definition of Lorenz domination, such as Atkinson [1970] and Dasgupta, Sen, and Starrett [1973], assumed that the compared profiles have the same mean, which does not fit in our setting. Therefore, our definition is an extension of their concept, which has referred to as generalized Lorenz dominance by Shorrocks [1980].

curves of first profile are nowhere lower than those of latter profile. It also indicates that a regular SWF is compatible with Lorenz domination criterion. Therefore, restricting attention on *regular* SWF to develop robust inequality index is plausible.

2.4 Two Robust Indices

In this section, we extend two of the most popular indices, namely, the Atkinson index and the Gini index, to the robust indices[6]. To discuss about the two specific indices, we need to restrict our robust inequality index I further. An inequality index I is a *relative* or scale invariant index if for all $F \in \mathcal{F}$ and $c > 0$, $I(cF) = I(F)$. To make I a relative index[7], further assumption on SWF W is required. We say W is *homothetic* if for all F, $W(F) = \Phi(\hat{W}(F))$, where $\hat{W}$ is linear homogeneous, *i.e.* $\hat{W}(cF) = c\hat{W}(F)$ for $c > 0$, and Φ is an increasing transformation.

Proposition 3. *A robust index of inequality I defined as in eq (1) is a relative index if and only if W is homothetic.*

Since the following indices we consider are relative, we restrict our attention to SWF that is both regular and homothetic.

Robust Atkinson index

We first consider a regular and homothetic SWF, so-called *robust Atkinson SWF*, which would characterize a robust Atkinson index, namely,

(2)
$$W_A(F) = \alpha \sum_{I=1}^{n} u(\overline{F}_i) + (1-\alpha) \sum_{i=1}^{n} u(\underline{F}_i).$$

where $0 \leq \alpha \leq 1$ and $u : \mathcal{X} \to \mathbb{R}$ is defined by

(3)
$$u(x) = \begin{cases} a + b \cdot \dfrac{x^r}{r} & \text{for } 0 < r < 1, \\ a + b \cdot \log x & \text{for } r = 0; \end{cases}$$

with constant number a and positive number b. Using the SWF above, we get the explicit form of

[6] We refer to chapter 2 of Moulin [1991] for a discussion of two classic indices developed on AKS approach.

[7] We refer to Blackorby and Donaldson [1980] for detailed discussion about relative index.

the *robust Atkinson index* according to eq (1):

$$I_A(F) = \begin{cases} 1-\alpha\Big[\frac{1}{n}\cdot\sum_{i=1}^{n}\big(\frac{\overline{F}_i}{\mu(\overline{F})}\big)^r\Big]^{1/r} - (1-\alpha)\Big[\frac{1}{n}\cdot\sum_{i=1}^{n}\big(\frac{\underline{F}_i}{\mu(\underline{F})}\big)^r\Big]^{1/r} & \text{for } 0<r<1, \\ 1-\alpha\Big[\prod_{i=1}^{n}\big(\frac{\overline{F}_i}{\mu(\overline{F})}\big)^{1/n}\Big] - (1-\alpha)\Big[\prod_{i=1}^{n}\big(\frac{\underline{F}_i}{\mu(\underline{F})}\big)^{1/n}\Big] & \text{for } r=0. \end{cases} \tag{4}$$

This index is a weighted average of indices of lower limit and upper limit distributions. The parameter α can be interpreted as the social confidence that upper limit distribution is the real distribution. Therefore, as α increases, society becomes more confident that upper limit distribution is the real distribution. The parameter r plays the similar role as in classic Atkinson index that r represents the degree of inequality aversion to transfers of allocation at different levels.

Robust Gini index

We now consider a SWF that characterizes a robust Gini index.

$$\begin{aligned} W_G(F) &= \alpha\Big\{\mu(\overline{F}) - \frac{\sum_{i=1}^n\sum_{j=1}^n|\overline{F}_i-\overline{F}_j|}{2n^2}\Big\} + (1-\alpha)\Big\{\mu(\underline{F}) - \frac{\sum_{i=1}^n\sum_{j=1}^n|\underline{F}_i-\underline{F}_j|}{2n^2}\Big\} \\ &= \alpha\cdot\frac{\sum_{i=1}^n\big[2(n-i)+1\big]\cdot\widetilde{\overline{F}}_i}{n^2} + (1-\alpha)\cdot\frac{\sum_{i=1}^n\big[2(n-i)+1\big]\cdot\widetilde{\underline{F}}_i}{n^2}, \end{aligned} \tag{5}$$

where $0 \le \alpha \le 1$. Hence, the robust Gini index defined below corresponds to the above SWF.

$$\begin{aligned} I_G(F) &= \alpha\cdot\frac{\sum_{i=1}^n\sum_{j=1}^n|\overline{F}_i-\overline{F}_j|}{2n^2\mu(\overline{F})} + (1-\alpha)\cdot\frac{\sum_{i=1}^n\sum_{j=1}^n|\underline{F}_i-\underline{F}_j|}{2n^2\mu(\underline{F})} \\ &= 1-\alpha\cdot\frac{\sum_{i=1}^n\big[2(n-i)+1\big]\cdot\widetilde{\overline{F}}_i}{n^2\mu(\overline{F})} - (1-\alpha)\cdot\frac{\sum_{i=1}^n\big[2(n-i)+1\big]\cdot\widetilde{\underline{F}}_i}{n^2\mu(\underline{F})} \end{aligned} \tag{6}$$

The Gini index might be the most widely used index of inequality and our robust Gini index provides a way to measure Gini index whenever allocation profile is not deterministic. The parameter α, once again, can be regarded as the confident weight that society assigns to upper limit distribution in a profile.

3 AXIOMATIZATION

In this section, we discuss the axioms that a society should satisfy in order to have a robust Atkinson or a robust Gini SWF. Based on the characterization, a transformation method we introduced in the previous section would lead to the robust Atkinson index and the robust Gini index. Formally, a social preference over a set of allocation profiles is denoted by $\succsim \subset \mathcal{F} \times \mathcal{F}$. We say a SWF $W : \mathcal{F} \to \mathbb{R}$ represents social preference $\succsim$ if for all $F, G \in \mathcal{F}$, $W(F) \geq W(G)$ if and only if $F \succsim G$.

3.1 Regular Axioms

We first state five regular axioms. These axioms with respect to deterministic profiles are widely assumed in the inequality literature. Also the five axioms are necessary for both robust Atkinson and robust Gini SWF.

A1 (*Weak order*) $\succsim$ is complete and transitive.

A2 (*Continuity*) For all $F \in \mathcal{F}$, the sets $\{G : G \succsim F\}$ and $\{G : F \succsim G\}$ are closed in $\mathcal{F}$ with respect to Hausdorff distance.

A1 is commonly required conditions and do not need further elaboration. A2 generalizes traditional continuity for deterministic profiles and can be interpreted in a similar manner.

For a permutation $\pi : \mathcal{N} \to \mathcal{N}$ and $F \in \mathcal{F}$, define $\pi \circ F \in \mathcal{F}$ by $(\pi \circ F)_i = F_{\pi(i)}$ for every $i \in \mathcal{N}$.

A3 (*Symmetry*) For all $F, G \in \mathcal{F}$, if there is a permutation π such that $F = \pi \circ G$, then $F \sim G$.

A3 says that every permutations of individual labels are regarded as allocation equivalent. It requires that the social ranking use only the information about the allocated variable and not about, for example, some other characteristic which might be discernible among society. Thus, according to symmetry, the identities of individuals are totally irrelevant to social decision process. Despite that it is not self-evident, this axiom is prevalently assumed in the literature.

A4 (*Unanimity*) For all $F, G \in \mathcal{F}$, if $F \geq G$, then $F \succsim G$.

A4 says that if each individual has higher allocations in F than in G, then society should prefer allocation profile F to G.

We say profile F *dominates* profile G if (i) for every $f \in F$, there exists a $g \in G$ such that $f \succsim g$, and (ii) for every $g \in G$, there exists $f \in F$ such that $f \succsim g$. In other words, if profile F dominates G, then for any deterministic allocation in F, there must exist a worse deterministic allocation in G; further, for any deterministic allocation in G, there must exist a better deterministic profile in F. The next axiom simply states that a dominant profile is always preferred to a dominated profile.

A5 (*Dominance*.) If one allocation profile F dominates another one G, then $F \succsim G$.

The above five axioms are intuitive assumptions in the inequality literature. Below we discuss further the very axioms that would characterize either robust Atkinson SWF or robust Gini SWF.

3.2 Robust Atkinson SWF

We now want to state the required axioms that characterize robust Atkinson SWF. To state next axiom, we need some notation first. If $F \in \mathcal{F}$ and $T \subset \mathcal{N}$, we write $F_T = (F_i)_{i \in T}$ and $F_{T^c} = (F_i)_{i \in \mathcal{N} \setminus T}$.

A6 (*Separability*) For all $F, G \in \mathcal{F}$ and nonempty $T \subset \mathcal{N}$, if $(F_T, F_{T^c}) \succsim (G_T, F_{T^c})$, then $(F_T, G_{T^c}) \succsim (G_T, G_{T^c})$

Separability basically says that when considering social welfare ordering, if two profiles only differ in subset T of individuals, then the allocation of T^c the rest individuals would not affect social ordering. In other words, social rankings are independent of nonconcerned individuals.

Along with the first four axioms, separability implies that social welfare function has a separably additive form, which is defined as below.

Definition 3. We say a SWF $W : \mathcal{F} \to \mathbb{R}$ is *separably additive* if there exist an increasing function $u : \mathcal{X} \to \mathbb{R}$ such that, for all $F \in \mathcal{F}$,

$$W(F) = \sum_{i=1}^{n} u(F_i).$$

Proposition 4. *A social preference $\succsim$ satisfies A1-4 and A6 if and only if there exists a separably additive SWF represents $\succsim$.*

This result says that a social preference that satisfies A1-4 and A6 is equivalent to the existence of a utility function defined on a set of possible allocation $\mathcal{X}$ such that any allocation profile is evaluated by the utility sum over every individual allocation. Furthermore, this utility function is increasing in $\mathcal{X}$. In contrast, the classic separably additive SWF is defined over deterministic allocation profile. Our result can be regarded as a direct extension of classic one.

Actually, robust Atkinson SWF is separably additive in which function u has the following form: there exists $\alpha \in [0, 1]$ such that for $Y \in \mathcal{X}$,

$$u(Y) = \alpha \max_{x \in Y} u(x) + (1 - \alpha) \min_{x \in Y} u(x).$$

Along with A5, the next axiom will characterize function u with the above expression. The last two axioms will guarantee function u on X has the expression as in eq (3).

For $Y \in \mathcal{X}$, we say an allocation $e(Y) \in X$ is *equivalent* to Y, if profile $(Y, \ldots, Y) \sim (e(Y), \ldots, e(Y))$. In words, if a profile has the same allocation Y for every individual, then a deterministic profile with allocation $e(Y)$ for every individual is socially equivalent.

A7 (*Commutativity.*) For $x_1, x_2, y_1, y_2 \in X$, if $x_1 \geq \{x_2, y_1\} \geq y_2$, then $F \sim G$ whenever $F_i = \{e(x_1, x_2), e(y_1, y_2)\}$ and $G_i = \{e(x_1, y_1), e(x_2, y_2)\}$ for all i.

To better understand the commutativity, see Figure (2) for the indifference curves over profiles $(Y, \ldots, Y)$ in which Y contains at most two values. Any point (x, y) in the quadrant represents profile $(Y, \ldots, Y)$ where $Y = \{x, y\}$. Therefore, the diagonal represents the deterministic profiles $(c, \ldots, c)$. Take any four possible allocation x_1, x_2, y_1, y_2, in which x_1 is the best allocation and y_2 is the worst allocation. Consider two allocation profiles. In the first profile, every individual allocations consist of the one equivalent to $\{x_1, x_2\}$ and the other one equivalent to $\{y_1, y_2\}$. In the other profile, every individual allocations consist of the one equivalent to $\{x_1, y_1\}$ and the other one equivalent to $\{x_2, y_2\}$. Then A7 requires that the two profiles are socially indifferent. In other words, for a so-defined allocation profile F, switching only the intermediate allocations x_2, y_1 would not change the social welfare ranking. Thus, in spirit A7 is a version of famous Thomason condition, which implies u is separably additive if Y contains at most two values.

A8 (*Scale Invariance*) For all deterministic profiles $f, g \in X^n$ and all $\lambda > 0$, if $f \succsim g$, then $\lambda f \succsim \lambda g$.

Under scale invariance axiom, it does not matter whether we measure allocation in euros or dollars as long as the unit is the same for each individual allocation.

A9 (*Pigou-Dalton principle*) For all deterministic profiles $f, g \in X^n$, if there are $i, j \in \mathcal{N}$ such that $f_k = g_k$ for $k \notin \{i, j\}$ and $f_i + f_j = g_i + g_j$ and $|f_i - f_j| < |g_i - g_j|$, then $f \succ g$.

A9 simply states that a transfer between two individual allocation, in such a way that their allocation difference is reduced, will result in a strictly social preferred allocation profile. This principle demonstrates that redistributions from the rich to the poor would improve the social welfare.

Theorem 1. *A social preference $\succsim$ on $\mathcal{F}$ satisfies A1-9 if and only if there exists a robust Atkinson SWF as in Eq (2) represents $\succsim$.*

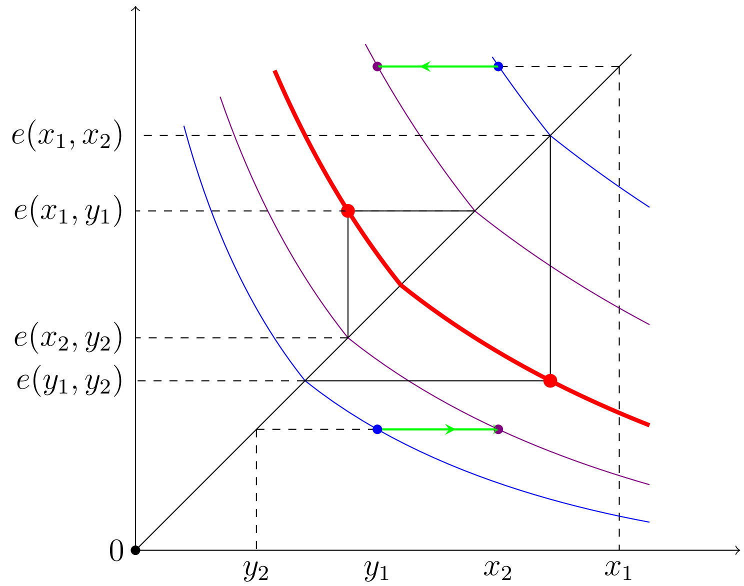

Figure 2: Commutativity

This result provides a characterization of robust Atkinson SWF when individual allocation may not be deterministic. Therefore, a social preference that respects A1-9 concerns the welfare loss due to inequality and imprecision in each allocation profile. Furthermore, a mathematical transformation of this SWF as in eq (1) induces a robust Atkinson index as in eq (4).

3.3 Robust Gini SWF

We now want to characterize robust Gini SWF. As we see from eq (5), robust Gini SWF is not separably additive. It is additive with respect to order-preserving. Formally, two deterministic allocation profiles $f, g \in X^n$ are *order-preserving* if $f_i \geq f_j \Leftrightarrow g_i \geq g_j$ for all $i, j \in \mathcal{N}$. For $F, G \in \mathcal{F}$, we say F and G are *order-preserving* (in boundary) if both $\overline{F}, \overline{G}$ and $\underline{F}, \underline{G}$ are order-preserving. For every F, G, we define $F + G$ by for each $i \in \mathcal{N}$,

$$(F + G)_i = \{f_i + g_i : f_i \in F_i \text{ and } g_i \in G_i\}.$$

Note that if F, G, H are pairwisely order-preserving profiles, then $F + H$ and $G + H$ are also pairwisely order-preserving.

A6' (*Order-preserving Independence.*) For all $F, G, H \in \mathcal{F}$, if F, G, H are pairwisely order-preserving, then $F \succsim G \Leftrightarrow F + H \succsim G + H$.

This axiom states that the social ranking of two profiles F and G, which agree on the ordering of upper and lower limits, respectively, should be invariant to the addition of another order-preserving profile H. The inspiration for it may best be seen through the cases it precludes: if, for instance, two profiles $F+H$ and $G+H$ are the addition of a common profile H; and F and G are not order-preserving, then the overall judgement between $F+H$ and $G+H$ is not completely determined by a comparison of F and G. Suppose individual i is the richest in F, but the poorest in G. On the contrary, individual j is the poorest in F, but the richest in G. If H is a profile with high allocation for i, but low allocation for j, then addition of F and H may make the difference between i and j even larger. As a result, profile $F+H$ is more unequal than F. At the same time, the addition of G and H would reduce the difference between i and j and is more equal than H. Therefore, it is not promising to insist the invariance to addition of the common profile. This asymmetric impact on inequality may give rise to preference reversal. A6' only requires that if the profiles are order-preserving, then preference reversal should not occur. Also, this axiom can be regarded as a generalization of traditional order-preserving independence over deterministic profiles (See Weymark [1981]).

We state the last three axioms to derive classic Gini SWF defined on deterministic profiles. The next two axioms are first proposed by Ben Porath and Gilboa [1994]. For $f \in X^n$ and $i, j \in \mathcal{N}$, we say i precedes j in f if $f_i \leq f_j$ and there is no $k \in \mathcal{N}$ such that $f_i < f_k < f_j$.

A7' (*Transfer Invariance.*) For all $f, g, f'g' \in X^n$ and $i, j \in \mathcal{N}$, if the following are satisfied:

- **(i)** i precedes j in $f, g, f'g'$;
- **(ii)** $f_i = f'_i + c$, $f_j = f'_j - c$ and $g_i = g'_i + c$, $g_j = g'_j - c$ for some $c > 0$;
- **(iii)** $f_k = f'_k$ and $g_k = g'_k$ for $k \notin \{i, j\}$,

then $f \succsim g$ if and only if $f' \succsim g'$.

A7' requires that there is no preference reversal if there is same amount of transfer between two preceded individuals i, j. However, it is indeed a strong claim since it is possible that i, j are poor in f, but rich in g.

A8' (*Inequality Aversion.*) For all $f, g \in X^n$ and $i \in \mathcal{N}$, if $\tilde{f}_i = \tilde{g}_i + c$ and $\tilde{f}_{i+1} = \tilde{g}_{i+1} - c$ for some $c > 0$ and $\tilde{f}_j = \tilde{g}_j$ for $j \notin \{i, i+1\}$, then $f \succ g$.

A8' simply says that it is socially preferred that if we transfer an amount of money from an individual to the next richest one without changing the ordering. This axiom is a weaker version of Dalton-Pigou principle, in which any transfer from rich to poor is preferred.

A9' (*Tradeoff.*) For all $c > 0$ and $k \in \mathcal{N}$,

$$(kc, 0, \ldots, 0) \sim (\underbrace{\frac{c}{k}, \ldots, \frac{c}{k}}_{k \text{ individuals}}, 0, \ldots, 0)$$

The intuition of A9' is the following. Suppose a society with total income c. If society transfers the total income to one individual, then he will have k times income return and the rest of society have nothing. Or society can divide total income equally among k individuals without any return and leave nothing to the rest of society. Tradeoff axiom requires that a society should be indifferent between two options. A9' illustrates precisely how a society balances between equity and efficiency. Society needs a geometric growth of income to compensate equality loss.

Theorem 2. *A social preference $\succsim$ on $\mathcal{F}$ satisfies A1-5 and A6'-9' if and only if there exists a robust Gini SWF as in eq (5) that represents $\succsim$.*

This result fully characterizes the robust Gini SWF. This SWF is not separably additive, but order-preserving additive. Note that $\succsim$ restricted to deterministic profiles is classic Gini SWF. However, our characterization improves that part of Ben Porath and Gilboa [1994] since their results are restricted to deterministic profiles with fixed total income. Aaberge [2001] suggests an axiomatic characterization of classic Gini SWF based on Lorenz curve orderings, which is initiated by Yaari [1988]. However, his result is built on the assumption that Lorenz cure is convex, which is not necessarily the case in our framework. Therefore, we provide the first fully characterization of classic Gini index as a by-product. In sum, if a society believes in the set of axioms we suggest above, then robust Gini index should be superior to the midpoint Gini index.

4 Concluding Remark

It is increasingly understood that inequality has impacted nearly every aspect of economics. Many inequality measurement studies carried out over the past several decades have provided a precise snapshot of inequality, under the assumption that each individual allocation can be precisely estimated. However, many widely used data only provide imprecise estimation, which bring both conceptual and practical challenges in measuring inequality. This paper first explores a novel development of measuring inequality in the face of indeterministic allocation profiles. According to this methodology, this paper extends the classic Atkinson and Gini indices to the robust ones. We then provide an axiomatic justification of those associated SWFs. This innovation corrects for some shortcomings of traditional treatment, while having some drawbacks of their own. We now provide a simple example to indicate its shortcomings.

A society consists of two individuals. Consider an allocation profile F in which $F_1 = F_2 = \{1, 10\}$. Therefore, $\overline{F} = (10, 10)$ and $\underline{F} = (1, 1)$. Clearly, the robust (Gini) inequality index we suggest is zero, which means the society is completely equal. However, we may not feel comfortable about this conclusion since we could not rule out the possible allocations $(1, 10)$ and $(10, 1)$, which seem quite unequal. This observation reveals that while measuring inequality, restricting only on upper and lower limit distributions are problematic. A more plausible measurement should account for every possible allocations in the very profile. An alternative robust inequality measure could be defined as follows: A function $J : \mathcal{F} \to \mathbb{R}$ is a *weighted maxmin Gini inequality index* if for all $F \in \mathcal{F}$,

$$J(F) = \alpha \max_{f \in F} I_g(f) + (1 - \alpha) \min_{h \in F} I_g(h),$$

where $0 \leq \alpha \leq 1$ and I_g is the classic Gini index . Index J represents the weighted average of highest inequality and lowest inequality in F. The parameter α captures the weight a society assigns to the least inequality. Consider again the above example, it is obvious that index J is strictly positive as long as $\alpha > 0$. Although the index J avoids the shortcomings robust index I has, it is not immediately clear how to derive the corresponding SWF. Continuing to improve upon these measures is important and needs more work on it.

APPENDIX: PROOFS

A PROOF OF SECTION 2

A.1 Proof of Proposition 1

To prove Proposition 1, suppose that function $I : \mathcal{F} \to \mathbb{R}$ is defined as in eq (1) and the associated SWF W is regular.

Proof of (i). By definition, for $F \in \mathcal{F}$,

$$I(F) = \alpha_F I(\overline{F}) + (1 - \alpha_F) I(\underline{F}).$$

Since $\alpha_F \in [0, 1]$, it is immediate to see that $\min\{I(\overline{F}), I(\underline{F})\} \leq I(F) \leq \max\{I(\overline{F}), I(\underline{F})\}$.

Proof of (ii). For $F \in \mathcal{F}$, $W(F) = W(\xi(F) \cdot \mathbb{1})$. By continuity of W, ξ is also a continuous function. By monotonicity of W, for any $F, G \in \mathcal{F}$,

$$W(F) \geq W(G) \Longleftrightarrow \xi(F) \geq \xi(G).$$

Since W is Schur concave with respect to deterministic profiles, for any bistochastic matrix M of order n,

$$W(fM) \geq W(f).$$

For any $c > 0$, $W(c \cdot \mathbb{1}) = c$. So, we have

$$W(fM) = \xi(fM) \geq W(f) = \xi(f).$$

Note that $\mu(fM) = \mu(f)$. Therefore,

$$\begin{aligned} I(fM) &= 1 - \frac{\xi(fM)}{\mu(fM)} \\ &= 1 - \frac{\xi(fM)}{\mu(f)} \\ &\leq 1 - \frac{\xi(f)}{\mu(f)} \\ &= I(f) \end{aligned}$$

Proof of (iii). Consider bistochastic matrix $\hat{M} = (m_{ij})$ with $m_{ij} = 1/n$ for all $i, j \in \mathcal{N}$. Then for any $f \in \mathcal{F}$, Schur concavity implies that

$$W(f\hat{M}) = \mu(f) \geq W(f).$$

Therefore, for $F \in \mathcal{F}$, we have

$$0 \leq \frac{\xi(\overline{F})}{\mu(\overline{F})}, \frac{\xi(\underline{F})}{\mu(\underline{F})} \leq 1.$$

Therefore, according to the definition of I,

$$0 \leq I(\overline{F}), I(\underline{F}) \leq 1.$$

So, for any $\alpha_F \in [0, 1]$,

$$I(F) = \alpha_F I(\overline{F}) + (1 - \alpha_F) I(\underline{F}) \in [0, 1].$$

Furthermore, if $I(F) = 0$, then $I(\overline{F}) = I(\underline{F}) = 0$. This means that $\xi(F) = \mu(F)$, which leads to $F = c \cdot \mathbb{1}$ for some $c > 0$. Conversely, if $F = c \cdot \mathbb{1}$, we have $\xi(F) = \mu(F)$, which leads to $I(F) = 0$.

A.2 Proof of Proposition 2

Since the proof of necessity part is straightforward, we only prove sufficiency part. Suppose $F \succsim_L G$. By definition, for all $f \in F$ and $g \in G$, $f \succsim_L g$. Lorenz dominance requires that for all $k = 1, 2, \ldots, n$

$$\frac{1}{n\mu(f)} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{1}{n\mu(g)} \sum_{i=1}^{k} \tilde{g}_i.$$

Since $\mu(\underline{F}) = \min_{f \in F} \mu(f) \geq \max_{g \in G} \mu(g) = \mu(\overline{G})$, we have

$$\frac{\mu(\underline{F})}{n\mu(f)} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{\mu(\overline{G})}{n\mu(g)} \sum_{i=1}^{k} \tilde{g}_i,$$

which implies

$$\frac{1}{n} \sum_{i=1}^{k} \tilde{f}_i \geq \frac{1}{n} \sum_{i=1}^{k} \tilde{g}_i.$$

Now, according to Marshall, Olkin, and Arnold [1979] (pp 64), there must exist a bistochastic matrix M such that $\tilde{f} \geq \tilde{g}M$. Then, monotonicity of W implies $W(\tilde{f}) \geq W(\tilde{g}M)$. Furthermore, Schur-concavity implies that $W(\tilde{g}M) \geq W(\tilde{g})$. Notice that Schur-concavity implies symmetry, hence, $W(\tilde{f}) = W(f)$ and $W(\tilde{g}) = W(g)$. As a result, we have $W(f) \geq W(g)$. Since this inequality holds for any $f \in F$ and $g \in G$,

$$\min_{f \in F} W(f) \geq \max_{g \in G} W(g).$$

Monotonicity requires that $W(F) \geq \min_{f \in F} W(f)$ and $\max_{g \in G} W(g) \geq W(G)$, which implies $W(F) \geq W(G)$.

A.3 Proof of Proposition 3

We first show the necessity part: suppose that I is a relative index. By definition, we have

$$\begin{aligned}\xi(F) &= \lambda_F \xi(\overline{F}) + (1-\lambda_F)\xi(\underline{F}) \\ &= \lambda_F \mu(\overline{F})(1 - I(\overline{F})) + (1-\lambda_F)\mu(\underline{F})(1 - I(\underline{F}))\end{aligned}$$

Since index I is homogeneous of degree zero, linear homogeneity of mean μ implies linear homogeneity of ξ.

$$\begin{aligned}W(F) &= W(\xi(F) \cdot \mathbb{1}) \\ &= \Phi(\xi(F)),\end{aligned}$$

where Φ is increasing in its argument. Hence, W is homothetic.

Now we show the sufficiency part: suppose that W is homothetic. Then, there exist an increasing function Φ and a linearly homogeneous function $\hat{W}$ such that for $F \in \mathcal{F}$,

$$W(F) = \Phi(\hat{W}(F)).$$

Since $\hat{W}$ is linearly homogeneous, we have

$$\xi(F) = \frac{\hat{W}(F)}{\hat{W}(\mathbb{1})}.$$

Therefore, ξ is also linearly homogeneous. Since μ is also linearly homogeneous, robust index I defined as above becomes homogeneous of degree zero. Thus, I is a relative index.

B Proof of Section 3

B.1 Proof of Proposition 4

The necessity part is straightforward. We only prove the sufficiency part. Suppose $\succsim$ satisfies A1-4 and A6.

First, restricted $\succsim$ to set of deterministic profiles X^n. Since X is connected and separable, and $\succsim$ satisfies conditions of Debreu [1960] separable Theorem, there exists a continuous function $u_i : X \to \mathbb{R}$ such that the sum of u_i represents $\succsim$. Symmetry further requires that each u_i has to be identical. Therefore, there is a continuous function $u : X \to \mathbb{R}$ such that $f \succsim g \Leftrightarrow \sum_{i=1}^{n} u(f_i) \geq$

$\sum_{i=1}^{n} u(g_i)$. Furthermore, A5 unanimity implies that u is also increasing in X.

Now, we extend u from domain X to $\mathcal{X}$ in the following way. For $Y \in \mathcal{X}$ and $c \in X$, we define $u(Y) = u(c)$ if $F \sim f$ whenever $F_i = Y$ and $f_i = c$ for all i. Since Y is compact, there exist $a, b \in X$ such that $a \geq Y \geq b$. Unanimity implies that equally distributed profiles must satisfy the preferences: $(a, \ldots, a) \succsim (Y, \ldots, Y) \succsim (b, \ldots, b)$. Therefore, by continuity, there exists a unique c such that $(c, \ldots, c) \sim (Y, \ldots, Y)$. Hence, u on $\mathcal{X}$ is well-defined.

Pick any $F = (Y_1, \ldots, Y_n) \in \mathcal{F}$. Let $c_1, \ldots, c_n$ in X be such that $u(Y_i) = u(c_i)$ for all i. To prove the additive separability, it suffices to show that $F \sim (c_1, \ldots, c_n)$. We prove it by induction.

Claim 1. *For any $i \in \mathcal{N}$, $(c_1, \ldots, c_{i-1}, Y_i, c_{i+1}, \ldots, c_n) \sim (c_1, c_2, \ldots, c_n)$.*

Proof of Claim: By A3 symmetry, it suffices to prove that $(Y_1, c_2, \ldots, c_n) \sim (c_1, \ldots, c_n)$. Furthermore, by separability, we only need to show the case where $(Y_1, c_1, \ldots, c_1) \sim (c_1, c_1 \ldots, c_1)$. Suppose such indifference relation does not hold. Assume first that

$$(Y_1, c_1, \ldots, c_1) \succ (c_1, \ldots, c_1).$$

Then, separability implies that $(Y_1, \ldots, Y_1) \succ (c_1, Y_1, \ldots, Y_1)$. According to definition, $(c_1, \ldots, c_1) \sim (Y_1, \ldots, Y_1)$, which implies that

$$(Y_1, c_1, \ldots, c_1) \succ (c_1, Y_1, \ldots, Y_1).$$

By symmetry, it is equivalent to $(c_1, Y_1, c_1, \ldots, c_1) \succ (c_1, Y_1, \ldots, Y_1)$. Applying separability again, we have

$$(Y_1, Y_1, c_1, \ldots, c_1) \succ (Y_1, \ldots, Y_1, Y_1) \succ (c_1, Y_1, \ldots, Y_1).$$

Similarly, we can use separability and symmetry again to get

$$(Y_1, Y_1, Y_1, c_1 \ldots, c_1) \succ (Y_1, \ldots, Y_1, Y_1) \succ (c_1, Y_1, \ldots, Y_1).$$

Repeat this process, we finally have $(Y_1, \ldots, Y_1, c_1) \succ (Y_1, \ldots, Y_1, Y_1)$, which contradicts to our assumption.

Now, if we assume the other possibility that $(c_1, \ldots, c_1) \succ (Y_1, c_1, \ldots, c_1)$, it is similar to show the contradiction. □

Claim 2. *If $(Y_1, \ldots, Y_t, c_{t+1}, \ldots, c_n) \sim (c_1, \ldots, c_n)$, then $(Y_1, \ldots, Y_{t+1}, c_{t+2}, \ldots, c_n) \sim (c_1, \ldots, c_n)$.*

Proof of Claim: By separability, it suffices to prove that if $(Y_1, \ldots, Y_t, c, \ldots, c) \sim (c_1, \ldots, c_t, c, \ldots, c)$ for some t, then it holds for $t + 1$. Since $(Y_1, \ldots, Y_t, c, \ldots, c) \sim (c_1, \ldots, c_t, c, \ldots, c)$, separability implies that

$$(Y_1, \dots, Y_{t+1}, c, \dots, c) \sim (c_1, \dots, c_t, Y_{t+1}, c, \dots, c).$$

By Claim 1, $(c_1, \dots, c_t, Y_{t+1}, c, \dots, c) \sim (c_1, \dots, c_{t+1}, c, \dots, c)$. Hence, this claim holds. $\square$

By Claim 1 and 2, for any $F \in \mathcal{F}$, we define $W : \mathcal{F} \to \mathbb{R}$ by $W(F) = \sum_{i=1}^{n} u(F_i)$, which clearly represents $\succsim$.

B.2 Proof of Theorem 1

Sufficiency Part:

Suppose that $\succsim$ on $\mathcal{F}$ satisfies A1-9. Our strategy to prove that robust Atkinson SWF represents $\succsim$ is following: First, we consider only the profiles that every individual have identical and binary values. We show that there exists unique $\alpha \in (0, 1)$ such that for any $x > y$ in X, $u(\{x, y\}) = \alpha u(x) + (1 - \alpha)u(y)$. Second, we consider the profiles that every individual have identical, but arbitrarily many outcomes. We show that for any $Y \in \mathcal{X}$, $u(Y) = \alpha u(\max_{x \in Y} x) + (1 - \alpha)u(\min_{y \in Y} y)$. Third, we show that A8 scale invariance and A9 Pigou-Dalton principle imply that u on X has either power function or log function form. Finally, combined with Proposition 4, A5 dominance implies that for any $F \in \mathcal{F}$,

$$W(F) = \alpha \sum_i u(\overline{F}_i) + (1 - \alpha) \sum_i u(\underline{F}_i)$$

represents $\succsim$.

To start, notice that proposition 4 implies the existence of u on $\mathcal{X}$. Define $\succsim^*$ on X^2 by

$$(a, b) \succsim^* (c, d) \Leftrightarrow u(\{a, b\}) \geq u(\{c, d\}).$$

Lemma B1. *For all $a, b, c \in X$, if $a \geq b$, then $(a, c) \succsim^* (b, c)$.*

Proof. Take $a, b, c \in X$ with $a \geq b$. Let $Y = \{a, c\}$ and $Z = \{b, c\}$. So profile $(Y, \dots, Y)$ dominates profile $(Z, \dots, Z)$. By A5, $(Y, \dots, Y) \succsim (Z, \dots, Z)$. Proposition 4 implies $u(Y) \geq u(Z)$. Hence, by definition, $(a, b) \succsim^* (b, c)$. $\square$

Let $0 < \ell \leq \ell' < \infty$. Consider $\succsim^*$ restricted to $[0, \ell] \times [\ell', +\infty)$. We show that this restricted preference has an additive conjoint structure, hence has a separably additive utility representation.

Lemma B2. $\succsim^*$ *restricted to $[0, \ell] \times [\ell', +\infty)$ satisfies the following conditions:*

A1* *(weak order):* $\succsim^*$ *is complete and transitive.*

A2* *(Independence):* $(x,b') \succsim^* (y,b')$ *implies* $(x,x') \succsim^* (y,x')$*; also,* $(b,x') \succsim^* (b,y')$ *implies* $(x,x') \succsim^* (x,y')$*.*

A3* *(Thomsen):* $(x,z') \sim^* (z,y')$ *and* $(z,x') \sim^* (y,z')$ *imply* $(x,x') \sim^* (y,y')$*.*

A4* *(Essential): There exist* $b,c \in [0,\ell]$ *and* $a \in [\ell',+\infty)$ *such that* $(b,a) \nsim^* (c,a)$*, and* $b' \in [0,\ell]$ *and* $a',c' \in [\ell',+\infty)$ *such that* $(b',a') \nsim^* (b',c')$*.*

A5* *(Solvability): If* $(x,x') \succsim^* (y,y') \succsim^* (z,x')$*, then there exist* $a \in [0,\ell]$ *such that* $(a,x') \sim (y,y')$*; if* $(x,x') \succsim^* (y,y') \succsim^* (x,z')$*, then there exists* $a' \in [\ell',+\infty)$ *such that* $(x,a') \sim^* (y,y')$*.*

A6* *(Archimedean): For all* $x,x' \in [0,\ell]$ *and* $y,z \in [\ell',+\infty)$*, if* $(x,y) \succsim^* (x',z)$*, then there exists* a,b *in* $[0,\ell]$ *satisfying* $(x,y) \succsim^* (a,y) \sim^* (b,z) \succ^* (b,y) \succsim^* (x',z)$*. A similar statement holds with the roles of* $[0,\ell]$ *and* $[\ell',+\infty)$ *reversed.*

Proof. By definition, $\succsim^*$ is a weak order. It is easy to show all the axioms except Thomsen condition. Below, we show Thomsen condition.

Suppose that $(x,z') \sim^* (z,y')$ and $(z,x') \sim^* (y,z')$. By definition, this is equivalent to $u(x,z') = u(z,y')$ and $u(z,x') = u(y,z')$. To show that $u(x,x') = u(y,y')$, there are three cases to consider: $z' \geq \{x',y'\}$, $y' \geq \{x',z'\}$ and $x' \geq \{y',z'\}$.

Suppose first that $z' \geq \{x',y'\}$. Since $\{x',y'\} \geq \{x,y,z\}$, Lemma B1 implies that $(x',z') \succsim^* (x,z')$ and $(y',z') \succsim^* (y,z')$. Thus, A7 commutativity implies

$$u(e(x,x'),e(z',z')) = u(e(x,z'),e(x',z')).$$

Note that $(x,z') \sim^* (z,y')$ implies $e(x,z') = e(z,y')$. Therefore,

$$u(e(x,z'),e(x',z')) = u(e(z,y'),e(x',z')).$$

Applying commutativity again, we have

$$u(e(z,y'),e(x',z')) = u(e(z,x'),e(y',z')).$$

Note again that $(z,x') \sim^* (y,z')$ implies $e(z,x') = e(y,z')$. Therefore,

$$u(e(z,x'),e(y',z')) = u(e(y,z'),e(y',z')).$$

Commutativity implies that

$$u(e(y,z'),e(y',z')) = u(e(y,y'),e(z',z')).$$

Therefore, we have $u(e(x,x'),e(z',z')) = u(e(y,y'),e(z',z'))$, which implies, by Lemma B1, $e(x,x') = e(y,y')$. That is, $u(x,x') = u(y,y')$.

For the other two cases, similar arguments will lead to the same results. □

Lemma B3. *There exist two real-valued functions ϕ and φ on X such that for all $x,x',y,y' \in X$ with $x \leq y$ and $x' \leq y'$,*

$$(x,y) \succsim^* (x',y') \Longleftrightarrow \phi(x)+\varphi(y) \geq \phi(x')+\varphi(y').$$

Furthermore, if there are ϕ',φ'represents $\succsim^*$ instead of ϕ,φ, respectively, then there exist $\gamma > 0$ and β_1,β_2 such that $\phi' = \gamma\phi+\beta_1$ and $\varphi' = \gamma\varphi+\beta_2$.

Proof. Let $a > 0$. Lemma B2 implies that $\succsim^*$ restricted to $[0,a]\times[a,+\infty)$ is an additive conjoint structure. Thus, by Theorem 2 of Chapter 6 in Krantz, Luce, Suppes, and Tversky [2006], there exist two function ϕ_a on $[0,a]$ and φ_a on $[a,+\infty)$ represent $\succsim^* \subset [0,a]\times[a,+\infty)$, *i.e.* for all $x,x' \in [0,a]$ and $y,y' \in [a,+\infty)$

$$(x,y) \succsim^* (x',y') \Longleftrightarrow \phi_a(x)+\varphi_a(y) \geq \phi_a(x')+\varphi_a(y').$$

By uniqueness of representation, we can normalize ϕ_a and φ_a such that

$$u(a) = \phi_a(a)+\varphi_a(a).$$

If $b > a$, since $\succsim^* \subset [0,b]\times[b,+\infty)$ is also an additive conjoint structure, then there exist functions ϕ_b on $[0,b]$ and φ_b on $[b,+\infty)$ that represent such preferences. Due to the uniqueness of representation, we can normalized ϕ_b in the way such that $\phi_b(a) = \phi_a(a)$. By similar method, if $c \in (0,a)$, since $\succsim^* \subset [0,c]\times[c,+\infty)$ is also an additive preference structure, then there exist functions ϕ_c on $[0,c]$ and φ_c on $[c,+\infty)$ that represent such preferences. Again, φ_c is normalized in the way that $\varphi_c(a) = \varphi_a(a)$.

Now, define $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \phi_x(x) & \text{if } x > 0; \\ \phi_a(0) & \text{if } x = 0. \end{cases}$$

Similarly, define $\varphi : X \to \mathbb{R}$ by

$$\varphi(y) = \begin{cases} \varphi_y(y) & \text{if } y > 0; \\ u(0) - \phi_a(0) & \text{if } y = 0. \end{cases}$$

Therefore, ϕ and φ on X are uniquely specified. According to continuity and unanimity, $\varphi(0) < \varphi(y)$ for all $y > 0$. Take arbitrary $0 < y \leq x$. There always exists a, b such that $x < a$ and $0 < b < y$. Therefore,

$$\begin{aligned} x \geq y &\Leftrightarrow (x, a) \succsim^* (y, a) \\ &\Leftrightarrow \phi_a(x) + \varphi_a(a) \geq \phi_a(y) + \phi_a(a) \\ &\Leftrightarrow \phi_x(x) \geq \phi_y(y) \\ &\Leftrightarrow \phi(x) \geq \phi(y) \end{aligned}$$

Similarly, we have

$$\begin{aligned} x \geq y &\Leftrightarrow (b, x) \succsim^* (b, y) \\ &\Leftrightarrow \phi_b(b) + \varphi_b(x) \geq \phi_b(b) + \phi_b(y) \\ &\Leftrightarrow \varphi_b(x) \geq \varphi_b(y) \\ &\Leftrightarrow \varphi(x) \geq \varphi(y) \end{aligned}$$

Therefore $x \geq y \Leftrightarrow \phi(x) + \varphi(x) \geq \phi(y) + \varphi(y)$. We show that ϕ and φ have the properties above. Let $x \leq y$ and $x' \leq y'$. Suppose that $(x, y) \succsim^* (x', y')$. There are two cases: either $x \geq y'$ or $x < y'$. First, assume that $x \geq y'$. Then, continuity and unanimity imply that there exists a and b such that $(x, y) \sim^* (a, a)$ and $(b, b) \sim^* (x', y')$.

$$\begin{aligned} (x, y) \sim^* (a, a) \Leftrightarrow \phi(x) + \varphi(y) = \phi(a) + \varphi(a), \\ (x', y') \sim^* (b, b) \Leftrightarrow \phi(x') + \varphi(y') = \phi(b) + \varphi(b). \end{aligned}$$

Note that $a \geq b$, which is $\phi(a) + \varphi(a) \geq \phi(b) + \varphi(b)$. Therefore,

$$(x, y) \succsim^* (x', y') \Leftrightarrow \phi(x) + \varphi(y) \geq \phi(x') + \varphi(y').$$

The uniqueness of representation follows immediately from the definition of ϕ and φ. $\square$

Lemma B4. *There exists* $0 \leq \alpha \leq 1$ *such that for all* $x \geq y$,

$$u(\{x, y\}) = \alpha u(x) + (1 - \alpha) u(y).$$

Proof. It suffices to show that there are constants $\beta > 0$ such that $\varphi(x) = \beta\phi(x)$. If $a > 0$, define

$$\phi_{1a}(x) = \phi(e(a, x)) \quad \text{and} \quad \varphi_{1a}(x) = \varphi(e(a, x)), \text{ for } x \geq a;$$
$$\phi_{2a}(x) = \phi(e(x, a)) \quad \text{and} \quad \varphi_{2a}(x) = \varphi(e(x, a)), \text{ for } x \leq a.$$

For $\{x, y, z, w\} \geq a$, if $x \leq y$ and $z \leq w$, then $(a, y) \succsim^* (a, x)$ and $(a, w) \succsim^* (a, z)$. Therefore,

$$\begin{aligned}(z, w) \succsim^* (x, y) &\Leftrightarrow \phi(z) + \varphi(w) \geq \phi(x) + \varphi(y) \\ &\Leftrightarrow e(z, w) \geq e(x, y) \\ &\Leftrightarrow (a, e(z, w)) \succsim^* (a, e(x, y))\end{aligned}$$

The last equivalence is implied by Lemma B1. Commutativity implies that $(a, e(z, w)) \sim^* (e(a, z), e(a, w))$ and $(a, e(x, y)) \sim^* (e(a, x), e(a, y))$. Therefore,

$$\begin{aligned}(z, w) \succsim^* (x, y) &\Leftrightarrow \phi(z) + \varphi(w) \geq \phi(x) + \varphi(y) \\ &\Leftrightarrow (e(a, z), e(a, w)) \succsim^* (e(a, x), e(a, y)) \\ &\Leftrightarrow \phi(e(a, z)) + \varphi(e(a, w)) \geq \phi(e(a, x)) + \varphi(e(a, y)) \\ &\Leftrightarrow \phi_{1a}(z) + \varphi_{1a}(w) \geq \phi_{1a}(x) + \varphi_{1a}(y).\end{aligned}$$

If $x \leq y \leq a$ and $z \leq w \leq a$, then similarly we have

$$(z, w) \succsim^* (x, y) \Leftrightarrow \phi_{2a}(z) + \varphi_{2a}(w) \geq \phi_{2a}(x) + \varphi_{2a}(y).$$

Thus ϕ_{1a} and φ_{2a} represent $\succsim^*$ on $[0, a] \times [a, +\infty)$. By uniqueness of representation, there are $k_1, k_2 > 0$ and k_{11} and k_{12} such that for $a, b > 0$,

$$\phi_{1a}(x) = k_1(a)\phi(x) + k_{11}(a) \quad \text{and} \quad \varphi_{2b}(y) = k_2(b)\varphi(y) + k_{12}(b).$$

Notice that if φ is constant, then it is trivially that $u(\{x, y\}) = \phi(x) = u(x)$, which is $\alpha = 1$. Similarly, if ϕ is constant, then $u(\{x, y\}) = u(y)$, which is $\alpha = 0$. Now, suppose both ϕ and φ are non-constant. Take $w \geq \{y, z\} \geq x$. Lemma B1 implies that $(z, w) \succsim^* (x, y)$ and $(y, w) \succsim^* (z, x)$.

According to commutativity,

$$\begin{aligned}
&(e(x,y),e(z,w)) \sim^* (e(x,z),e(y,w)) \\
&\Leftrightarrow \phi(e(x,y)) + \varphi(e(z,w)) = \phi(e(x,z)) + \varphi(e(y,w)) \\
&\Leftrightarrow k_1(x)\phi(y) + k_{11}(x) + k_2(w)\varphi(z) + k_{12}(w) = k_1(x)\phi(z) + k_{11}(x) + k_2(w)\varphi(y) + k_{12}(w) \\
&\Leftrightarrow k_1(x)(\phi(y) - \phi(z)) = k_2(w)(\varphi(y) - \varphi(z)).
\end{aligned}$$

Since the above equations are satisfied for all x, y, z, w with $w \geq \{y, z\} \geq x$, there exist positive constants λ, δ such that

$$k_1(x) = \lambda \quad \text{and} \quad k_2(y) = \delta.$$

Thus, for all $y, z > 0$,

$$\lambda(\phi(y) - \phi(z)) = \delta(\varphi(y) - \varphi(z)).$$

Hence, there are $\beta > 0$ such that $\phi(x) = \beta\varphi(x)$ for all x. Let $\alpha = \frac{1}{1+\beta}$. Clearly $0 < \alpha < 1$. According to unique representation, we can normalize $u(x) = \frac{\phi(x)}{\alpha}$. Therefore, for $x \leq y$,

$$\phi(x) + \varphi(y) = \alpha u(x) + (1-\alpha)u(y).$$

□

Lemma B5. *There exist $a \in \mathbb{R}$ and $b > 0$ such that for every $x \in X$,*

$$u(x) = \begin{cases} a + b \cdot \dfrac{x^r}{r} & \textit{for } 0 < r < 1 \\ a + b \cdot \log x & \textit{for } r = 0. \end{cases}$$

Proof. Restricted $\succsim$ to deterministic profiles. Since $\succsim$ is continuous and separable on X^n, Roberts [1980] demonstrates that scale invariance implies that function u has the following forms: there are constant a and positive b such that

$$u(x) = \begin{cases} a + b \cdot \dfrac{x^r}{r} & \text{for } r > 0 \\ a - b \cdot \dfrac{x^r}{r} & \text{for } r < 0 \\ a + b \cdot \log x & \text{for } r = 0. \end{cases}$$

Note that Pigou-Dalton principle means that for $x, y, z, w \in X$, if $x + y = z + w$ and $|x - y| <$

$|z-w|$, then $u(x)+u(y) \geq u(z)+u(w)$. This is equivalent to for all $x<y$ and all $c>0$

$$u(x+c)-u(x) \geq u(y+c)-u(y),$$

which implies that u is concave on X. Thus, concavity of u requires that $r \leq 1$. Furthermore, unanimity requires that $r \geq 0$. Therefore, u must have the expression stated at this lemma. $\square$

For $Y \in \mathcal{X}$, denote $y^* = \max_{y \in Y} y$ and $y_* = \min_{y \in Y} y$.

Lemma B6. *For $Y \in \mathcal{X}$, $u(Y) = u(y^*, y_*)$.*

Proof. Take $Y \in \mathcal{X}$. Since $\{y^*, y_*\} \subseteq Y$, we know $(Y, \ldots, Y)$ dominates $(\{y^*, y_*\}, \ldots, \{y^*, y_*\})$. By definition of y^* and y_*, it is immediate that $(\{y^*, y_*\}, \ldots, \{y^*, y_*\})$ also dominates $(Y, \ldots, Y)$. Therefore, according to dominance axiom, $(\{y^*, y_*\}, \ldots, \{y^*, y_*\}) \sim (Y, \ldots, Y)$. This is equivalent to $u(y^*, y_*) = u(Y)$. $\square$

Necessity Part:

Suppose that $\succsim$ is represented by a robust Atkinson SWF W. We want to prove that this preference satisfies A1-9. We only demonstrate commutativity axiom since the rest axioms are straightforward.

Consider $x_1, x_2, y_1, y_2 \in X$ where $x_1 \geq \{x_2, y_1\} \geq y_2$. Let $F \in \mathcal{F}$ be such that $F_i = \{e(x_1, x_2), e(y_1, y_2)\}$ for all i. Also, let $G \in \mathcal{F}$ be such that $G_i = \{e(x_1, y_1), e(x_2, y_2)\}$ for all i. According to the representation function, we have

$$\begin{aligned}
u(e(x_1,x_2), e(y_1,y_2)) &= \alpha u(x_1,x_2) + (1-\alpha)u(y_1,y_2) \\
&= \alpha[\alpha u(x_1) + (1-\alpha)u(x_2)] + (1-\alpha)[\alpha u(y_1) + (1-\alpha)u(y_2)] \\
&= \alpha[\alpha u(x_1) + (1-\alpha)u(y_1)] + (1-\alpha)[\alpha u(x_2) + (1-\alpha)u(y_2)] \\
&= \alpha u(x_1,y_1) + (1-\alpha)u(x_2,y_2) \\
&= u(e(x_1,y_1), e(x_2,y_2)).
\end{aligned}$$

B.3 Proof of Theorem 2

Since the necessity part is straightforward, we only show the sufficiency part. Suppose that $\succsim$ satisfies A1-5 and A6'-9'. Our strategy is first to show that $\succsim$ restricted to deterministic profile have Gini SWF. Then we show that if $\succsim$ restricted to the profiles in which individual 1 has binary values and all the rest individuals have singleton value, then $\succsim$ has a robust Gini SWF representation. Finally, we extend this result to the whole set of profiles.

Lemma C1. *Let $\succsim$ restrict to X^n. Then there exists ϕ on X^n such that*

$$\phi(f) = \mu(f) - \frac{\sum_i \sum_j |f_i - f_j|}{2n^2},$$

represents $\succsim$ on X^n.

Proof. It is clear to see that $\succsim$ on X_c^n also satisfies A1-4 and A6'-8'. Therefore, according to Theorem D of Ben Porath and Gilboa [1994], there exist $0 < \delta < \frac{1}{n(n-1)}$ and ϕ on X^n such that for $f \in X^n$

$$\phi(f) = \mu(f) - \delta \cdot \sum_i \sum_j |f_i - f_j|,$$

represents $\succsim$ on X^n. Pick $c > 0$ and $k \in \mathcal{N}$. By A9' tradeoff, we know $(kc, 0, \dots, 0) \sim (c/k, \dots, c/k, 0, \dots, 0)$. The above ϕ function implies that

$$\frac{kc}{n} - \delta(n-1)kc = \frac{c}{n} - \delta \cdot 2k(n-k)\frac{c}{k}.$$

Therefore, the only solution is

$$\delta = \frac{1}{2n^2}.$$

□

We denote

$$\mathcal{F}^1 = \{F \in \mathcal{F} : F_1 = \{a, b\} \text{ and } F_i = \{c\}, \forall i \neq 1 \text{ and } a, b, c \in X \text{ with } a, b \geq c\}.$$

the set of all profiles in which individual 1 is the richest with two possible allocations in the society, and the rest in the society have deterministic and equalized allocation. Note that for $F \in \mathcal{F}^1$, if $a = b$, then F is a deterministic profile; and if $a = b = c$, then F is a deterministic equally distributed profile.

Lemma C2. *If $F, G \in \mathcal{F}^1$, then F and G are order-preserving.*

Proof. This follows immediately from the definition of order-preserving. □

Lemma C3. *For $F, f \in \mathcal{F}^1$, if $F \sim f$, then $\alpha F + (1-\alpha) f \sim f$ for all $\alpha \in (0, 1)$.*

Proof. Pick $F \in \mathcal{F}^1$ be such that $F_1 = \{a, b\}$, $F_i = \{c\}$ for $i \neq 1$ and $a \geq b \geq c$. If there is a deterministic profile $f \in \mathcal{F}^1$ be such that $F \sim f$, we should have $\frac{F}{2} \sim \frac{f}{2}$. To see this, suppose not.

Assume that $\frac{F}{2} \succ \frac{f}{2}$. Since $\frac{F}{2}, \frac{f}{2} \in \mathcal{F}^1$, A6' order-preserving independence implies that

$$\frac{F}{2} + \frac{F}{2} \succ \frac{F}{2} + \frac{f}{2} \succ \frac{f}{2} + \frac{f}{2} = f.$$

Notice that

$$(\frac{F}{2} + \frac{F}{2})_1 = \{a, \frac{a+b}{2}, b\} \quad \text{and} \quad (\frac{F}{2} + \frac{F}{2})_i = c.$$

Recall that the representation of $\succsim$ restricted on deterministic profile can also be written as

$$\phi(f) = \frac{1}{n^2} \sum_{i=1}^{n} [(2(n-k)+1]\tilde{f}_i$$

Therefore, $\overline{F} = (a, c, \ldots, c)$ is the most preferred deterministic profile in both F and $\frac{F}{2} + \frac{F}{2}$, i.e.

$$\overline{F} = (a, c, \ldots, c) \in \arg\max_{f \in F} \phi(f) \qquad \text{and} \qquad \overline{F} = (a, c, \ldots, c) \in \arg\max_{f \in \frac{F}{2} + \frac{F}{2}} \phi(f);$$

and $\underline{F} = (b, c, \ldots, c)$ is the least preferred deterministic profile in both F and $\frac{F}{2} + \frac{F}{2}$. Hence, F and $\frac{F}{2} + \frac{F}{2}$ dominates each other. According to A6', $F \sim \frac{F}{2} + \frac{F}{2}$, which contradicts the assumption that $F \sim f$. Now assume that $\frac{f}{2} \succ \frac{F}{2}$. We repeat the similar process as above and lead to a contradiction. Hence, $F \sim f$ implies $\frac{F}{2} \sim \frac{f}{2}$.

Proceeding with induction, we have for every integer $k = 1, 2, \ldots$

$$\frac{F}{k} \sim \frac{f}{k}.$$

Also, by A6',

$$F \sim f \Rightarrow F + F \sim F + f \sim f + f = 2f.$$

Observe that $(2F)_1 = \{2a, 2b\}$, $(F+F)_1 = \{2a, a+b, 2b\}$ and $(2F)_i = (F+F)_i = \{2c\}$ for $i \neq 1$. Since $a \geq b$, it is immediate that $2F$ and $F+F$ dominates each other, therefore, $2F \sim F+F$. Hence $F \sim f$ implies $2F \sim 2f$. By induction, we have for every integer k,

$$kF = kf.$$

Combine the results abover, for every positive rational number α, we have

$$\alpha F \sim \alpha f.$$

Continuity implies that the above result holds for every positive real number α. Now, take any $\alpha \in (0,1)$ and apply A6',

$$\alpha F \sim \alpha f \Leftrightarrow \alpha F + (1-\alpha) f \sim f.$$

□

Recall that for $F \in \mathcal{F}$, $\overline{F}$ and $\underline{F}$ represents the upper limit and lower limit distribution in F, respectively.

Lemma C4. *There exists $\alpha \in [0,1]$ such that for all $F \in \mathcal{F}^1$,*

$$F \sim \alpha \overline{F} + (1-\alpha)\underline{F}.$$

Proof. If $x \in X$, we define $\mathcal{F}^1(x) = \{F \in \mathcal{F}^1 : F_i = \{x\} \text{ for all } i \neq 1\}$ denote the collection of profiles in $\mathcal{F}^1$ in which, except individual 1, every individual have equal allocation. Therefore, $\mathcal{F}^1 = \cup_{x \in \mathbb{R}} \mathcal{F}(x)$. We first show that the result holds on the restricted domain $\mathcal{F}^1(0)$.

Referring to Figure 3. For $f \in \mathcal{F}^1(0)$ with $F_1 = \{a,b\}$, F can be identified by the point (a,b) if $a > b$. Similarly, for $F \in \mathcal{F}^1(0)$ with $F_1 = \{c\}$, F can be identified by the point (c,c). Therefore, there is one-to-one correspondence between set $\mathcal{F}^1(0)$ and the points between horizontal axis and diagonal. For every $F, G \in \mathcal{F}^1(0)$, where $F_1 = \{a,b\}$ and $G_1 = \{c,d\}$, we define

$$(a,b) \succsim (c,d) \Leftrightarrow F \succsim G.$$

Take $a > b$. We have

$$(a,a) \succ (b,b).$$

By definition, we know that profile (a,a) dominates (a,b) and (a,b) dominates (b,b). Therefore, A6' implies

$$(a,a) \succsim (a,b) \succsim (b,b).$$

Continuity implies that there exists $\alpha \in [0,1]$ such that

$$(\alpha a + (1-\alpha)b, \alpha a + (1-\alpha)b) \sim (a,b).$$

Let $\alpha a + (1-\alpha)b = c$. Lemma C3 implies that any points on the straight line between (c,c) and (a,b) are indifferent. Therefore, every indifferent curve must be a straight line.

Now, we need to show that every indifferent lines parallel to each other. Take any point (a', b'). Connect points (a', b') and $(0,0)$ by a straight line. Without loss of generality, suppose this line

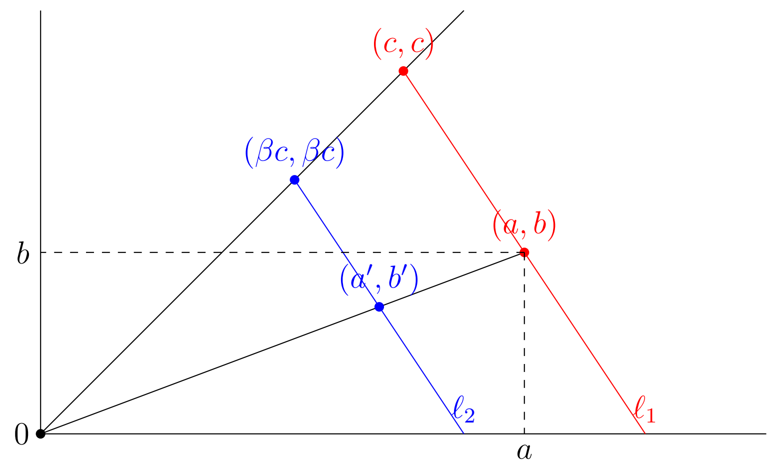

Figure 3: Indifference curve on $\mathcal{F}^1(0)$.

intersects the indifference curve, line between (c, c) and (a, b), at point (a, b). Therefore, there exists unique $\beta > 0$ such that

$$(a', b') = (\beta a, \beta b).$$

Since $(a, b) \sim (c, c)$, Lemma C3 implies that

$$(\beta a, \beta b) \sim (\beta c, \beta c).$$

Therefore, $(a', b') \sim (\beta c, \beta c)$, which means that two indifferent curves ℓ_1, ℓ_2 parallel to each other.

To finish our proof, we now extend the result from domain $\mathcal{F}^1(0)$ to $\mathcal{F}^1$. Pick any a, b, c such that $a \geq b \geq c > 0$. Consider a profile $F \in \mathcal{F}^1$ being such that $F_1 = \{a - c, b - c\}$ and $F_i = \{0\}$ for $i \neq 1$. Clearly, such F belongs to $\mathcal{F}^1(0)$ and, therefore,

$$(a - c, b - c) \sim (\alpha(a - c) + (1 - \alpha)(b - c), \alpha(a - c) + (1 - \alpha)(b - c)).$$

Now, adding constant deterministic profile $(c, \ldots, c)$ on both proifles, A6' implies that

$$F \sim (\alpha a + (1 - \alpha)b, c, \ldots, c).$$

Since $\overline{F} = (a, c, \ldots, c)$ and $\underline{F} = (b, c \ldots, c)$, we have $F \sim \alpha\overline{F} + (1 - \alpha)\underline{F}$. $\square$

We now define a real valued function W on $\mathcal{F}^1$ by, for $F \in \mathcal{F}$,

$$W(F) = \phi(\alpha\overline{F} + (1 - \alpha)\underline{F}).$$

It is immediate to see that W represents $\succsim$ restricted on $\mathcal{F}^1$. Notice that for each $F \in \mathcal{F}$, $\overline{F}$ and $\underline{F}$ are order-preserving. By the order-preserving additivity and homogeneity of ϕ, we have

$$W(F) = \alpha\phi(\overline{F}) + (1-\alpha)\phi(\underline{F})$$

represents $\succsim$ on $\mathcal{F}^1$.

Lemma C5. *For $F \in \mathcal{F}$ and $G \in \mathcal{F}^1$, if $\overline{F} \sim \overline{G}$ and $\underline{F} \sim \underline{G}$, then $F \sim G$.*

Proof. Since both F and G dominate each other, it is immediate that $F \sim G$ according to A5. $\square$

Now, we can extend real-valued function W to the whole set $\mathcal{F}$ by for $F \in \mathcal{F}$ if there is $G \in \mathcal{F}^1$ such that $\overline{F} \sim \overline{G}$ and $\underline{F} \sim \underline{G}$, then

$$W(F) = \alpha\phi(\overline{F}) + (1-\alpha)\phi(\underline{F}).$$

We claim that W represents the $\succsim$ on $\mathcal{F}$. To see this, note that by continuity, for every $F \in \mathcal{F}$, there must exist $F^1 \in \mathcal{F}^1$ such that $\overline{F} \sim \overline{F^1}$ and $\underline{F} \sim \underline{F^1}$. Take any $F, G \in \mathcal{F}$. According to Lemma C5, we have

$$\begin{aligned} F \succsim G &\iff F^1 \succsim G^1 \\ &\iff W(F^1) \geq W(G^1) \\ &\iff \alpha\phi(\overline{F^1}) + (1-\alpha)\phi(\underline{F^1}) \geq \alpha\phi(\overline{G^1}) + (1-\alpha)\phi(\underline{G^1}) \\ &\iff \alpha\phi(\overline{F}) + (1-\alpha)\phi(\underline{F}) \succsim \alpha\phi(\overline{G}) + (1-\alpha)\phi(\underline{G}) \\ &\iff W(F) \geq W(G). \end{aligned}$$

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