Perfect Altruism Breeds Time Consistency

Antoine Billot, Xiangyu Qu

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Abstract

Public policies should be analyzed through social lifetime utility. This paper focuses on the general process, namely, aggregation rules, that makes these policies socially acceptable to individuals through their own discount factors and instantaneous utilities. We show that *perfect altruism* via an adapted form of unanimity is the key condition helping to characterize a time-consistent social planner concerned with intergenerational fairness in the presence of individuals who are heterogeneous in discount factors and instantaneous utilities. In addition, different intensity levels of altruism are proven to provide different forms of aggregated social discounting and instantaneous utility, these forms giving rise to several lifetime utilities, from the standard exponential discounted function to the quasi-hyperbolic and the k-hyperbolic functions. Moreover, by demonstrating that the degree of social present bias can be regulated by the choice of the number of periods involving altruism through unanimity, new insights emerge and potentially overturn some of the most standard economic policy recommendations.

1 INTRODUCTION

Household savings, consumption, education or health insurance arbitrage ensues from very common decisions that involve a difficult process combining the need to cluster the heterogeneous interests of those who form the household with that of making a unique collective decision. Indeed, this challenge concerns issues far beyond household decisions. In fact, basic heterogeneity in lifetime preferences, over time discounting and instantaneous utility, affects most socioeconomic

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decisions, ranging from fiscal policy (Barro [1974]) to environmental policy (Nordhaus [2007]). For instance, the debate over the Stern Report essentially revolves around the choice of the *right* social discount factor to assume for the evaluation of long-term policies.

The heterogeneity of individual preferences is a priori expected to be reflected in the heterogeneity of impatience rates *as well as* the heterogeneity of instantaneous utilities. Certainly, from Marglin [1963] and Feldstein [1964] to Gollier and Zeckhauser [2005], the literature is broad and explains and illustrates how it is difficult to jointly derive aggregate discount factors and utility for a planner facing a society of heterogeneous agents. However, to be theoretically relevant, social time preferences should have a normative basis, and the most difficult aspect of this aggregation background is the twofold dimensional heterogeneity since this crucial twin heterogeneity is highlighted in numerous contributions. On the one hand, surveys by Frederick, Loewenstein, and O'Donoghue [2002] and more recently by Cohen, Ericson, Laibson, and White [2020] show that individual discount factors differ dramatically across different studies and estimations. Weitzman [2001] and Drupp, Freeman, Groom, and Nesje [2018], more recently, report that there is no natural convergence toward a unique impatience rate even among experts. On the other hand, it is a deeply established tradition in economic theory that individuals differ in their instantaneous utilities and, therefore, cannot be readily considered homogeneous. Neglecting no aspect of individual heterogeneity is unequivocally a target of our paper.

The usual approach to circumvent this heterogeneity in the case of intertemporal problems consists of treating the aggregate choices as if they were produced by a representative agent[1] whose instantaneous utility and impatience rate can be used as a representation of social instantaneous utility and a measure of social impatience. The *exponential discounted utility* (EDU) model due to Ramsey [1928] and Samuelson [1937] has long been recognized as the canonical model of this representative agent. Although Marglin [1963] and Feldstein [1964] have highlighted the difficulty of deriving a social lifetime EDU by aggregating a society of heterogeneous individuals, this form is still widely used to evaluate various policies because of its irresistible simplicity and elegance. The issue of the form of social discounting, however, has recently introduced novel challenges at the academic frontier between theoretical considerations and policy debates. One of these challenges is normative and has arisen in the context of climate change. Optimal climate policy is related to the social value of the future and, therefore, depends critically on the discounting factor (Nordhaus [2007]). Initiated by Ramsey's ethical critique in support of a near-one discounting factor, many studies have thus proposed that the social planner should impose a higher discount factor than that of the current generation (Bernheim [1989], Farhi and Werning [2007], Caplin and

[1]Here, to our minds, there is no conceptual difference between a society, a representative agent and a social planner.

Leahy [2004]). Nevertheless, this approach is built on the assumption that a dynastic individual and the planner discount the future differently. This entails both a conceptual and a theoretical difficulty in justifying this difference through a consistent preference aggregation process. Another challenge is descriptive and has arisen in the context of political power rotation. It is well known that political turnover leads to time inconsistency, which descriptively falsifies the social EDU assumption (Harstad [2020]). Although the potential implications of social time inconsistency have been frequently noted, few studies have formalized the mechanisms under which preference aggregation may lead to, for instance, quasi-hyperbolic discounting. This constitutes another target of our contribution.

More generally, the approach of this paper seeks to contribute to the literature on time preferences in several ways. We jointly characterize the social discount factor and instantaneous utility across two settings: a general *time-separable utility* (TSU) setting where individuals have TSUs and an EDU setting where individuals are fitted with EDUs. We identify the conditions to quantify social entities through parameters by aggregating individual entities in a nondictatorial fashion, namely, when every individual lifetime utility influences the formation of social utility. Specifically, we advocate the 'utilitarian' idea that society should take a weighted average of individual discount factors, which stands in stark contrast to the argument that society should value the future more than individuals.[2]

The economic tradition advises justifying the transition from individual to social entities by means of an aggregation rule and imposing that this rule satisfies the Pareto principle. As noted by Zuber [2011] and Jackson and Yariv [2014], this clearly contexts to the literature devoted to preference aggregation.[3] However, it is well known that a possible aggregation result becomes impossible when individuals are too heterogeneous. In terms of time preferences, the standard Pareto condition (PC) is not sufficient to withstand the effect of the heterogeneity of individual discount factors when individual instantaneous utilities are supposed to be heterogeneous and consequently to provide an axiomatic justification of social time preferences.

Intuitively, preference unanimity can result from the fact that conflicts over individual instantaneous utilities and conflicts over individual discounting factors cancel out in a TSU. We therefore suggest an alternative condition, the so-called impartial Pareto condition (IPC), which states that if all individuals rank one consumption stream higher than another, even when individual discount

[2] Weitzman [2001] and Drupp, Freeman, Groom, and Nesje [2018] contemplate some individuals who express support for a near-one discount factor. A social planner can, therefore, place a high weight on higher discounting factors and maintain intergenerational ethical concerns for long-run projects. Moreover, this weighted average, i.e., utilitarian, method is flexible enough to accommodate the demand for mild discounting in the case of short-run projects.

[3] In both cases, however, they are more interested in a dictatorial planner than in utilitarianism.

factors are impartially and arbitrarily permuted, then society should endorse this ranking. This new condition is significant: if both society and individuals have TSUs, then adopting the IPC yields a social discount factor and a social instantaneous utility that are equal to a weighted average of individual discount factors and individual utilities, respectively.

Next, the EDU model is generally regarded as a cornerstone for policy studies, thus demanding principles supporting its theoretical feasibility. We thus consider principles from the perspective of a society whose preferences are represented by an EDU and show first that, given individual preference heterogeneity, social lifetime utility may be dictatorial even under the IPC. Next, we find that a *perfectly altruistic* planner, that is, a planner who is only altruistic towards the next generation, as initiated by Phelps and Pollak [1968], is compatible with a social EDU. More precisely, the IPC must be accordingly adjusted to compare consumption streams that only differ in the same two periods (i.e., 2-IPC). We therefore show that a social TSU satisfying 2-IPC and a condition of stationarity must be a utilitarian EDU: social entities are identified as weighted means of associated individual entities. One prominent insight of this result is that perfect altruism, which drives time consistency, is shown to simply correspond to altruism between any two generations and not necessarily between two successive generations nor between the current and the next generations, as prescribed by Barro [1974]. Applications of this result to intergenerational transfer schemes, for instance, could be relevant.

Finally, as noted in Chambers and Echenique [2018] or Weitzman [2001], in many empirical situations, for instance, worldwide or European summits, board or household members, many decision-makers who can be assimilated to planners behave in a time-inconsistent way. Hence, we propose to study the extent to which deviating from perfect altruism would affect social discounting and, consequently, the time consistency of social lifetime utilities. The building block of this analysis is, perhaps surprisingly, that if a social TSU respects a 3-period IPC (3-IPC) corresponding to the occurrence of imperfect altruism and a stationarity-like condition, then the social discount factor is that of the quasi-hyperbolic discounting model (Phelps and Pollak [1968], Laibson [1997]). More interesting, we find that when the number of periods involving the IPC increases, society is more present biased. This suggests that the degree of social present bias can be regulated by controlling the number of periods involving the IPC. We do not search for an abstract specification of the optimal number of periods involving unanimity. Rather, in empirical situations, a social planner can be assumed to be nondogmatic, as in Millner [2020], which means that he must choose the very principle, i.e., the appropriate number of periods involving the IPC, in accordance with the problem at hand. In technology policy, for instance, Harstad [2020] stresses that time inconsistency and strategic investments are important for policies addressing externalities. Thus, once an optimal degree of time inconsistency is determined, the planner can select the associated number of periods involving the IPC to match this inconsistency.

Related Literature

In two different settings, Zuber [2011] and Jackson and Yariv [2014] show that a constant discounting society that respects the PC cannot aggregate individual lifetime preferences in a non-dictatorial manner if individual discount factors and instantaneous utilities are heterogeneous. For our part, even if we consider a setting quite similar to Jackson and Yariv's, we show in Theorem 1 that a nondictatorial aggregation is possible if the social lifetime utility satisfies the IPC, which is weaker than the PC. However, if all individuals are constant discounters, a separate aggregation rule results in present-biased social lifetime utility. Since constant discounting is a simple and tractable assumption for policy making, we argue in Theorem 1 that if a constant discounting society respects a slightly modified IPC, i.e., restricted to pairs of streams only differing in two periods, then the social discount factor is a weighted average of individual factors.[4] It has been observed by Phelps and Pollak [1968], Barro [1974], Kimball [1987], Saez-Marti and Weibull [2005], and more recently by Galperti and Strulovici [2017] that altruism with respect to the immediate generation would lead to time consistency. Although our issue and setting are substantially different from theirs, the fundamental insight we obtain is that such time consistency can be derived from altruism between two arbitrary generations, namely, not necessarily between two consecutive generations.

Chambers and Echenique [2018] literally disregard the heterogeneity of individual instantaneous utilities and suggest three aggregation rules for discount factors. One of them proposes aggregation by means of a weighted average method, which can then be viewed as an alternative approach to a special case of our Theorem 2. However, due to the significant difference between the two settings, the visions conveyed by the respective social principles are fundamentally different. By contrast, Feng and Ke [2018] argue that preferences of successive generations should be counted. Hence, they suggest an intergenerational PC and characterize a constant social discount factor that is greater than any individual factor.[5] Chichilnisky, Hammond, and Stern [2020] consider an extinction threat for future generations and accordingly propose 'extinction' social discounting. There are many other approaches to studying social time consistency. Millner and Heal [2018] demonstrates that a society can be time consistent if the assumption that social consumption

[4]This result can be regarded as an axiomatization of the exponential social discounter. The first axiomatization of this kind of discounting behavior is due to Koopmans [1960]. It was subsequently extended by Fishburn and Rubinstein [1982] and Bleichrodt, Rohde, and Wakker [2008] in a different way.

[5]Drugeon and Wigniolle [2020] studies a similar collective decision problem by assuming hyperbolic discounting individuals.

is time invariant is dropped. The same exercise, but in a continuous-time setting, is conducted by Drouhin [2020].

Although constant social discounting is an irresistible form, it is nevertheless rarely observed in policy making. In reality, either the institutional rotation of political power (see, Harstad [2020]) or the cost for commitment (see, Laibson [2015]) would be responsible for triggering present-biased policies. To understand the underlying behavioral mechanism, we show in Theorem 3 that if the social lifetime utility satisfies the IPC for any pair of consumption streams that are different in the first 3 periods, i.e., satisfies 3-IPC, then, along with some mild assumptions, this social lifetime utility admits a quasi-hyperbolic discounting form as in Phelps and Pollak [1968], Laibson [2015] and many others.[6] In fact, Gollier and Zeckhauser [2005] and Jackson and Yariv [2014] show that a social lifetime utility satisfying the PC is present biased if individuals have heterogeneous discount factors. Therefore, our Theorem 4 generalizes this observation in showing that various versions of the IPC may characterize comparatively different levels of present bias. This result relates to Millner [2020], where it is argued that a society may feel insecure under various normative arguments. Theorem 4 can then be interpreted as an axiomatic judgment about multiple social principles. In contrast to the 'present bias' approach, recent papers by Gonzalez, Lazkano, and Smulders [2018] and Ray [2018] show that society may exhibit future bias if there is a conflict of interest among future generations.

To the best of our knowledge, we are the first to identify the social principles necessary to obtain a separate aggregation of discount factors and instantaneous utilities. As Jackson and Yariv [2015] note, TSU is quite analogous to subjective expected utility, namely, we can interpret time as states and a discount function as a probability distribution over these states. In this regard, our result, by relaxing the PC to avoid Jackson-Yariv's and Zuber's impossibilities and obtaining separate aggregation, is conceptually related to Gilboa, Samet, and Schmeidler [2004] and Billot and Qu [2020], who show that a relaxed PC leads to separately aggregate heterogeneous beliefs and tastes. However, the issues are drastically different. In particular, the rule of aggregation for heterogeneous discount factors is fundamentally different from those used in a belief aggregation context since the nature and the measure of the two respective notions, a belief and an impatience rate, are not alike. Therefore, the respective results cannot draw lessons directly from one another.

The remainder of this paper proceeds as follows. Section 2 sets up the benchmark model. Section 3 motivates and formally states the IPC. Section 4 presents the separate aggregation results when individual utilities are TSUs, while Section 5 considers a society composed of EDU individ-

[6] This result can be regarded as an axiomatization of the quasi-hyperbolic social discounter. The axiomatization of this kind of behavior can be found in Hayashi [2003], Attema, Bleichrodt, Rohde, and Wakker [2010], Noor [2009] and Montiel Olea and Strzalecki [2014].

uals. Then, the characterization results of social time consistency and time inconsistency are both presented. Section 6 summarizes our findings and concludes the paper. All proofs are contained in the Appendix.

2 The Model

We consider a finite society $\mathcal{I}$ of n individuals i. Each individual is assumed to live infinitely and to consume in discrete periods $t \in \mathbb{N} = \{1, 2, \ldots\}$. Let $\mathcal{L}$ be the natural consumption space, formally a connected and separable topological space. Each t-period consumption z_t belongs to $\mathcal{L}$, and a *stream of consumption* is denoted by $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$. For any $z \in \mathcal{L}$, the *constant* consumption stream $(z, z, \ldots)$ is denoted by $\bar{\mathbf{z}}$. For any $x, y \in \mathcal{L}$ and $\mathbf{z} \in \mathcal{L}^\infty$, the particular consumption streams $(x, \mathbf{z})$ and $(x, y, \mathbf{z})$ denote $(x, z_1, z_2, \ldots)$ and $(x, y, z_1, z_2, \ldots)$, respectively. More generally, for any $t \in \mathbb{N}$ and any $\mathbf{x}, \mathbf{z} \in \mathcal{L}^\infty$, the stream $\mathbf{x}_t\mathbf{z}$ denotes $(x_1, \ldots, x_t, z_1, \ldots)$.

Individual preferences over alternative streams of consumption are represented by a lifetime utility function $U_i : \mathcal{L}^\infty \to \mathbb{R}$. We assume that such preferences are represented by *TSU*.[7] Namely, for each $t \in \mathbb{N}$ and each $i \in \mathcal{I}$, there exists an individual time weight or i's *discount factor at time* t denoted by $d_{it} > 0$ and a nonconstant and continuous *instantaneous utility* denoted by $u_i : \mathcal{L} \to \mathbb{R}$ such that a consumption stream $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$ is evaluated as follows:

$$U_i(\mathbf{z}) = \sum_{t=1}^{\infty} d_{it} u_i(z_t). \tag{1}$$

In a TSU model, d_{it} depends on time but not on consumption. Wlog, we normalize $d_{i1} = 1$, for all i. Positive discount factors reflect individual desirabilities of future consumption. Denote by $D_{\mathcal{I}}$ the set of all individual factors. Similarly, denote by $U_{\mathcal{I}}$ the set of all instantaneous individual utilities.

We assume two conditions. First, we assume the existence of a *minimum agreement over consumption* (MAC), i.e., there are $z_*, z^* \in \mathcal{L}$ such that, for $z \in \mathcal{L}$, $u_i(z_*) \le u_i(z) \le u_i(z^*)$, for all $i \in \mathcal{I}$. Second, we assume that social preferences over streams of consumption are also represented by a TSU. That is, there exists a continuous social instantaneous utility u and social

[7] In fact, TSU, whether individual or social, is implicitly assumed to depend only on relative time and flow variables but not on absolute time, i.e., it is *time invariant* in the sense of Halevy [2015]. History-dependent lifetime utility would then be an example violating time separability.

discount factor $d_t > 0$ such that the social lifetime utility function $U : \mathcal{L}^\infty \to \mathbb{R}$ is defined by:

$$U(\mathbf{z}) = \sum_{t=1}^{\infty} d_t u(z_t). \tag{2}$$

The TSU representation is the most general model of preferences satisfying time separability. This model is commonly used for both normative applications (prescribing optimal policy) and positive applications (describing and predicting behavior). The TSU model includes the hyperbolic discounting model where $d_t = (1 + \gamma t)^{-\frac{\alpha}{\gamma}}$ and $\alpha > \gamma$, the quasi-hyperbolic discounting model where $d_t = \beta\delta^{t-1}$, for $t > 1$ and many others.

The most important case of a TSU, which we will further discuss below, is the *exponential time discounted utility* (EDU). When preferences are TSU and satisfy Koopmans [1960]'s axioms, they can be represented by an EDU. Namely, for $i \in \mathcal{I}$, there exists a *constant* discount factor $\delta_i \in (0, 1)$ and a nonconstant and continuous instantaneous utility function $u_i : \mathcal{L} \to \mathbb{R}$ such that a consumption stream $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$ is evaluated by i as follows:

$$U_i(\mathbf{z}) = \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(z_t). \tag{3}$$

By extension, the triplet (U, δ, u) fully characterizes a social EDU.

A final remark concerns framework selection. We can alternatively consider a rich structure where consumption is defined as a lottery. This would have somewhat simplified our analysis and obtained the same results. However, we believe that lottery consumption is not a natural assumption to make in this context and, therefore, has no applicability in practice. It would also make the comparison with related studies less clear.

3 Pareto Dilemma and Impartiality

The standard PC was long widely accepted and considered an indisputable benchmark principle for preference aggregation. However, as suggested by Zuber [2011] and Jackson and Yariv [2015], among many others, the PC is a source of dilemmas in the dynamic preference aggregation setting. Indeed, when individual and social preferences are assumed to be represented by an EDU, the PC implies society to be dictatorial. Therefore, to better motivate the necessity of resorting to an alternative PC, we first demonstrate that even in our framework, where the lifetime utility of both individuals and society is assumed to be a TSU, the PC and nondictatorship are mutually exclusive.

3.1 Pareto Dilemma

When individual preferences and social preferences are supposed to be represented by a TSU, the PC can be written in the following way.

Pareto Condition (PC). For any $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_i(\mathbf{z}) \geq U_i(\hat{\mathbf{z}})$, for all $i \in \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

The PC means that if every individual prefers one consumption stream to another, then so does society. Unfortunately, although Buchanan and Tullock [1962] claim that it is ethically superior to all alternative principles, the PC is basically inconsistent with a nondictatorship requirement, which is commonly regarded as the minimum imperative for democracy.

To a large extent, society displays a degree of heterogeneity through, for instance, the heterogeneity of individual instantaneous utilities and time preferences. A society can then be said *regular* if (i) there are $i, j \in \mathcal{I}$ such that $d_{it} \neq d_{jt}$, for some t, and (ii) there exists $J \subseteq \mathcal{I}$ such that the subset U_J defined as $(u_i)_{i \in J}$ is the maximal linearly independent subset of individual instantaneous utilities.[8] Finally, a society is said to be *dictatorial* if there exists $i \in \mathcal{I}$ such that $U = U_i$.

Proposition 1. *Assume individual and social preferences to be represented by a lifetime TSU. Assume society to be regular. Then, the* PC *holds if and only if* U *is dictatorial.*

Proposition 1 basically means that when individuals are heterogeneous, the Pareto principle of unanimity is equivalent to the existence of a dictator. Although our setting is quite similar to that of Jackson and Yariv [2014], their result regarding the inconsistency between utilitarian aggregation and nondictatorship is different from ours. Despite some technical details,[9] they assume individual lifetime utilities to be EDUs and overall instantaneous utilities to be identical, i.e., it is assumed that there already exists a collective instantaneous utility function u. Their strategy consists then of adapting Harsanyi [1955]'s result to the field of time preferences while assuming that individual instantaneous utilities are previously aggregated (which implicitly amounts to assuming that instantaneous utilities are all identical). Consequently, they restrict the origin of the individual heterogeneity to only discount factor heterogeneity. However, this modeling option, even as a simplification, can hardly be viewed as empirically relevant. How can one reasonably postulate all experts' instantaneous utilities to be identical to justify collective decisions in combating climate change? In contrast, Proposition 1 introduces considerably more flexibility in generalizing Jackson and Yariv [2014]'s negative result to a larger class of individual lifetime utilities, i.e., all

[8] Recall that U_J is a linearly independent subset if $\sum_i \lambda_i u_i = 0$ implies $\lambda_i = 0$, for all $i \in J$.

[9] For example, in Jackson and Yariv [2014], the consumption space is only one dimensional, and instantaneous utilities are supposed to be twice differentiable.

TSUs, and, overall, to the case of heterogeneity of instantaneous utilities. However, for the sake of completeness, the second equivalence proved by Jackson and Yariv [2014], i.e., the equivalence between utilitarianism and present bias, is no longer valid in our setting, even in the case of a social TSU.

3.2 Fictitious Individuals and Impartial Pareto Condition

Before presenting the IPC, we propose a simple example to question the legitimacy of the PC. Consider a household consisting of two individuals, Ana, who is characterized by (u_a, d_{at}), and Bob, who is characterized by (u_b, d_{bt}). If this household is not dictatorial, then wlog, there exists a $\lambda \in (0, 1)$ such that

$$u = \lambda u_a + (1 - \lambda)u_b.$$

This household wants to decide whether to have a child. If they do not have a child, then their consumption in each period is constant, i.e., $(x, x, \cdots)$. If they have a child, then their first-period consumption y stands for consumption with 'baby child' and the second-period consumption z stands for consumption with 'adult child'. From the third period, consumption is constant, i.e., equal to x. Therefore, the consumption stream with a child is $(y, z, x, x, \cdots)$. Their instantaneous utility and relative discounting function are presented in the following table:[10]

$\mathcal{L}$	x	y	z
u_a	0	$\frac{0.98}{\lambda}$	$-\frac{1}{\lambda}$
u_b	0	$-\frac{0.95}{1-\lambda}$	$\frac{9}{1-\lambda}$
u	0	0.03	8

t	1	2
d_a	1	0.99
d_b	1	0.1
d	1	d_2

Ana enjoys the time with 'baby child' but worries about the future of 'adult child'. As a result, she deems y positive and z negative. However, Bob finds it boring and expensive to care for 'baby child' but enjoys family happiness once the child grows up and becomes old. As a result, he deems y negative and z positive. Furthermore, Ana is highly patient and has a low value for second-period consumption. In contrast, Bob is very impatient and highly values second-period consumption. By simple calculation, we have:

$$\begin{aligned} U_a(y, z, x, x, \cdots) &= \frac{0.98}{\lambda} - \frac{1}{\lambda} \times 0.99 < 0 = U_a(x, x, \cdots), \\ U_b(y, z, x, x, \cdots) &= -\frac{0.95}{1-\lambda} + \frac{9}{1-\lambda} \times 0.1 < 0 = U_b(x, x, \cdots). \end{aligned}$$

[10] Since instantaneous utility after the second period is always null, values of the discount function after the second period do not affect the calculation.

It is straightforward to see that both Ana and Bob prefer not to have a child. However, for any positive household discount function d, we have

$$U(y, z, x, x, \cdots) = 0.03 + 8d_2 > 0 = U(x, x, \cdots).$$

Therefore, *regardless of the discount function*, this household should have a child. This contradiction between the decision coming from individual preferences aggregated through the PC and that from the household lifetime preferences reveals that the current unanimity is *spurious*[11], i.e., Bob and Ana agree for opposite reasons.[12] This situation also reveals that unanimity as formalized through the PC violates the social interest of the household and hence can hardly be adopted by the household as a righteous principle. Intuitively, to avoid such spurious unanimity, both Ana and Bob should introduce some sympathetic or empathetic considerations. Ana (resp. Bob) should place herself (himself) in Bob's (Ana's) place and, to that end, replace her (his) discount function with Bob's (Ana's). If unanimity remains even while exchanging discount factors, then unanimity is no longer spurious but rather *impartial* insofar as no individual position is favored, and therefore, unanimity can be considered righteous.

More generally, concerning the PC, there are at least three kinds of shortcomings. First, the PC does not need to rest on unanimous reasons — it does not need to consider unanimous preferences as reflecting the existence of *reaching common ground*. As shown by the example above, agreed-on positions between Ana and Bob result from a tradeoff between patience rates and instantaneous utilities, i.e., an arbitration of contradictory interests. Such unanimity without common ground can hardly be seen as a compelling device for social decisions. This problem is prevalent in the general preference aggregation literature. Second, and more specifically, unanimous preferences do not ensure stationarity. An agreed-on position today does not necessarily hold tomorrow. The PC considers the need for current unanimity but ignores potential future 'temporal' conflicts and does not emphasize the necessity of future unanimity. Finally, empirical validation of the PC is limited. Although observable, social behaviors are unable to provide conclusive evidence for or against the PC.

As advocated in Billot and Qu [2020], a convincing PC should be rooted in the mutual acceptance of diverging opinions. This acceptance can be translated through a 'speculative' experience consisting of each individual placing himself in someone else's place. In other words, in terms of time preferences, a preference can be considered genuinely unanimous only if, when all individuals accept experimentation with the discount factor of other individuals belonging to the same society,

[11] This notion first appears in Mongin [1995].

[12] See also the 'duel example' in Gilboa, Samet, and Schmeidler [2004].

i.e., to replace their own factor with any other factor, this permutation never involves any preference reversal. This absence of preference reversal then reveals that such a speculative unanimity is robust to any individual discount factor, i.e., *impartial*, and, therefore, accommodates common ground to reach agreement.

Formally, any individual i can be regarded by society $\mathcal{I}$ as a formal couple $(d_i, u_i) \in D_{\mathcal{I}} \times U_{\mathcal{I}}$. Suppose that, in this couple, society (or a planner) replaces u_i with the instantaneous utility u_j of another individual belonging to $\mathcal{I}$ — in this case, j. Since this 'half-breed' individual ij composed of i's discount factor and j's instantaneous utility corresponds to a non-actual individual, he is basically *fictitious*. Note that only 'fictitious' individuals $ii \in \mathcal{I} \times \mathcal{I}$, for all $i \in \mathcal{I}$, are the 'true' individuals in society. Here, 'fiction' is an introspective experiment involving the association of a discount factor and an instantaneous utility that are not jointly observable in actual society. Namely, there is no actual individual corresponding to this formal couple. Then, call *fictitious society* the product set $D_{\mathcal{I}} \times U_{\mathcal{I}}$, i.e., for convenience $\mathcal{I} \times \mathcal{I}$. Assume now that the preferences of any fictitious individual $ij \in \mathcal{I} \times \mathcal{I}$ over streams of consumption are also represented by a TSU, i.e., $U_{ij} : \mathcal{L}^\infty \to \mathbb{R}$, where the t discount factor is d_{it} and the instantaneous utility is u_j:

$$U_{ij}(\mathbf{z}) = \sum_{t=1}^{\infty} d_{it} u_j(z_t).$$

The form U_{ij} expresses how individual j evaluates alternative consumption streams if she replaces her own discount factor with that of *any* individual i.[13]

Let us now introduce a modified PC that takes all fictitious individuals into account.

Impartial Pareto Condition (IPC). For any $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\hat{\mathbf{z}})$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

This modified PC means that, for each pair of consumption streams, if all fictitious individuals unanimously prefer one stream to the other, then so does society. In some ways, the IPC recalls the 'impartial observer' principle of Harsanyi [1953] in that it claims to neutralize the effects of individual heterogeneity by equating this heterogeneity with reaching common ground as the supposed foundation for unanimity. A simple way to understand Harsanyi's intuition about impartiality is the following: to help choose among social alternatives, each individual is assumed to imagine himself as an impartial observer who does not know which person he will be. Consequently, the impartial observer faces not only an actual distribution over the social outcomes but also what is

[13] Intuitively, in the last sentence, the pronoun '*any*' translates impartiality to the extent that society does not advantage a subset of individuals but requires considering any of them.

called ‘hypothetical lotteries’, i.e., ‘fictitious’ distributions. Hence, the set of hypothetical probabilities in Harsanyi’s and that of individual discount factors in our framework play an equivalent role and can be similarly interpreted.

When comparing the IPC to the PC, one critical difference arises. Under the IPC, society builds unanimous preferences from all possible fictitious preferences, not just actual preferences. Each individual is required to reevaluate every stream based on other individual discount factors to ensure unanimity to be fully compelling. Impartial introspection, i.e., considering the discount factor of anyone else as a possible introspective experience for oneself, can effectively eliminate the spurious unanimity induced by a double disagreement of instantaneous utilities and time preferences.

4 Separate Aggregation of Time Preferences

Two features are determined through preference aggregation: the social time discount function and the social instantaneous utility. Since we exclusively consider separate aggregation rules, the social time discount function is assumed to rely only on individual discount functions. The same holds for social instantaneous utility. Separate aggregation rules out, for instance, the case of consumption-dependent time discount functions. In the following theorem, the IPC is proven to provide a possibility for separate aggregation.

Theorem 1. *A social lifetime utility U satisfies the* IPC *if and only if there exist nonnegative $\{\alpha_i\}_{i\in\mathcal{I}}$ and $\{\gamma_i\}_{i\in\mathcal{I}}$ with $\sum_i \alpha_i = \sum_i \gamma_i = 1$ such that*

$$u = \sum_i \alpha_i u_i \quad \textit{and} \quad d_t = \sum_i \gamma_i \, d_{it} \tag{4}$$

for all $t \in \mathbb{N}$.

Theorem 1 states that if a social lifetime utility satisfies the IPC, social functions (utility and discount) take the form of a convex combination of individual functions. In contrast to impossibility results such as Proposition 1 and that of Jackson and Yariv [2014], the IPC weakens the PC in a way that avoids spurious unanimity. Hence, it naturally gives rise to a possibility. To see how the IPC works, note that it requires unanimity with respect to the *fictitious society*, which implies that there exists a nonnegative λ_{ij} such that, for a consumption stream $\mathbf{z}$,

$$U(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}(\mathbf{z}).$$

Let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, it can be shown that separate aggregation (4) holds.

This result does not help to determine any particular form of a normatively or descriptively appealing social discount function. However, we focus on two of most important types of discount functions: the *constant-impatience discount function* and the *present-bias discount function*, translating decreasing impatience. (We henceforth use the terms 'decreasing impatience' and 'present bias' interchangeably since it is widely agreed that the first serves as a testable implication of the second.) The key property of the first class of discount functions is that social choices are then always *time consistent*, which is not only normatively plausible but also widely applicable due to its tractability. This motivates the following question: is there a way of characterizing a constant-impatience social discount function? Nonetheless, one of the consequences of the existence of present bias is *time inconsistency*, which is not normatively plausible. However, at least since Thaler [1981], the finding that decision-makers become present biased as the time delay increases is a canonically descriptive result. Therefore, one might wonder what a social criterion driving such a present bias looks like.[14] Consequently, we attempt to characterize whether this criterion, while generating a social discount function, defines (dynamically inconsistent) decreasingly impatient behavior based on a reasonable definition of present bias. Such a criterion is expected, allowing not only the identification of discount functions but also the clarification of the key behavioral principles behind the collective decision-making process.

Let $(x_t, \bar{\mathbf{z}}_{*-t})$ denote a consumption stream with $z_t = x$ and $z_s = z_*$ for $s \neq t$.

Definition 1. A lifetime utility $U : \mathcal{L} \to \mathbb{R}$ is *present biased* (resp. *constant impatient*) if, for any $t > s$, any $k \geq 1$, $U(x_t, \bar{\mathbf{z}}_{*-t}) = U(y_s, \bar{\mathbf{z}}_{*-s})$ implies $U(x_{t+k}, \bar{\mathbf{z}}_{*-(t+k)}) \geq U(y_{s+k}, \bar{\mathbf{z}}_{*-(s+k)})$ (resp. $U(x_{t+k}, \bar{\mathbf{z}}_{*-(t+k)}) = U(y_{s+k}, \bar{\mathbf{z}}_{*-(s+k)})$).

For convenience, we allow ourselves to use 'constant impatient' as shorthand for 'translating a constant impatience'. A lifetime utility is present biased if, once closer consumption x at time s and further consumption y at time t are indifferent, then further consumption y is preferred when both consumption streams are shifted further by time k. Intuitively, if such time shifting does not change preferences, then this lifetime utility is said to be constant impatient.

Next, a present-biased U can also be characterized by its discount function. A discount factor measured at date t, i.e., $\delta(t)$, of a TSU with a discounting function d_t is defined as

$$\delta(t) = \frac{d_{t+1}}{d_t}.$$

The following lemma expresses that if a lifetime utility is present biased, then its discount factor is

[14] For the relation between social decreasing impatience and social present bias, see Jackson and Yariv [2015].

increasing. Similarly, if a lifetime utility is constant impatient, then its discount factor is constant.[15] This result is stated without proof because of its triviality.

Lemma 1. *Suppose that a lifetime utility U is a TSU characterized by (u, d_t). Then, U is present biased if and only if its discount factor δ is increasing. Moreover, U is constant impatient if and only if its discount factor δ is constant.*

Since we exclusively consider two types of lifetime utility functions, a natural question to ask is the following: is a society composed of constant-impatient or present-biased individuals and with lifetime preferences governed by the IPC necessarily present biased? The next proposition provides a positive answer to this question.

Proposition 2. *Assume that a social lifetime utility U satisfies the* IPC. *If each individual is either constant impatient or present biased, then a nondictatorial social utility U is necessarily present biased.*

Proposition 2 can also be viewed as a result displaying that a society composed of constant-impatient or present-biased individuals generates by separate aggregation of individual preferences a social lifetime utility that associates nondictatorship with present bias. This also proves that if the domain of individual lifetime utilities is restricted to the only constant-impatient or present-biased TSUs, the IPC implies that society is also present biased. Contrary to Jackson and Yariv [2015], this result thereby establishes that when a society is present biased, it is a social feature that does not rely on the assumption of constant-impatient individuals. Moreover, a present-biased society does not imply individuals to be either constant impatient or present biased. A simple example could be easily constructed with a first individual being present biased and a second individual increasingly impatient. If society assigns a small enough weight to the latter, society can still be present biased.

5 Stationarity, Altruism and Social Impatience

In this section, all individual preferences are assumed to be represented by an EDU. However, substantial empirical evidence supports that individuals do not behave as EDU maximizers when making decisions involving tradeoffs over time. Indeed, as noted by Frederick, Loewenstein, and O'Donoghue [2002], one generally assumes other kinds of behaviors that are more realistic, such as hyperbolic discounting. However, since our setting essentially revolves around common goods,

[15] In the case of an EDU, $d_{t+1}/d_t = \delta^t/\delta^{t-1} = \delta$.

individual preferences might very well differ from those concerning private goods. Furthermore, it is not clear why hyperbolic discounting behavior for private consumption should inform the assumption of discounting for common goods in the time horizon.

5.1 Stationarity, Perfect Altruism, and Constant Social Discounting

Although separate aggregation is compatible with the IPC, there also remains the question of the extent to which a dynamically inconsistent, i.e., present-biased, society can be collectively 'rational'. Since "*the simplicity and elegance of this* (EDU) *formulation is irresistible*", as claimed by Frederick, Loewenstein, and O'Donoghue [2002], it is of vital importance to suggest a principle that would characterize a society admitting an EDU representation for its preferences. In this subsection, we show that an 'appropriately modified' IPC along with a stationarity property would imply a time-consistent society that has instantaneous utility and discount factor defined as the convex combination of individual utilities and the convex combination of individual discount factors, respectively.

Consider first the stationarity axiom, which is required to ensure constant discounting, as shown by Koopmans [1960].

Stationarity. A lifetime utility function U is *stationary* if, for all $x \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) \geq U(\mathbf{z}') \text{ if and only if } U(x, \mathbf{z}) \geq U(x, \mathbf{z}').$$

Stationarity means that the ranking between two streams remains unchanged when common consumption is inserted in the first period for both streams. A decision-maker who obeys this axiom should be insensitive to what consumption is inserted. Recursively, it requires that the evaluation of two consumption streams does not change if all dates are shifted according to the same time constant.

However, the IPC and stationarity are not sufficient to characterize constant discounting for a nondictatorial social lifetime utility.[16] Therefore, a further weakened PC is needed to derive constant social discounting.

2-first-periods Impartial Pareto Condition (2-IPC). For any $x, y, x', y' \in \mathcal{L}$ and any $\mathbf{z} \in \mathcal{L}^\infty$, if $U_{ij}(x, y, \mathbf{z}) \geq U_{ij}(x', y', \mathbf{z})$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(x, y, \mathbf{z}) \geq U(x', y', \mathbf{z})$.

[16] Note that the IPC is equivalent to the PC when individual instantaneous utilities are identical. Jackson and Yariv [2015] demonstrate that there does not exist a nondictatorial social lifetime EDU if the PC and stationarity are imposed on a society where individuals have heterogeneous discount factors.

2-IPC states that when comparing two consumption streams that differ only in their first two consecutive periods, if all true individuals and all fictitious individuals prefer the first to the second stream, then so does society.

Phelps and Pollak [1968] recall that the intuition whereby individual preferences may be linked by a kind of 'generational' commitment already exists in Ramsey [1928], who assumes that each generation's preferences for its own consumption relative to the next generation's preferences do not differ from preferences for any future generation's consumption relative to that of the next generation. This commitment is equivalent to a stationarity postulate: the present generation's preferences over consumption streams are supposed to be invariant to changes in their timing. Phelps and Pollak [1968] suggest calling this *perfect altruism*. Later, Barro [1974]'s analysis of debt neutrality is based on a similar assumption: individuals are motivated by a special form of intergenerational altruism (here called *dynastic altruism*) such that individuals have an altruistic concern for their children, who in turn also have altruistic feelings for their own children, and so forth.[17]

While assimilating a period to a generation length, the restriction imposed by 2-IPC for time consistency is precisely to avoid imperfect altruism between generations, since imperfect altruism leads to a violation of stationarity. By considering only the first two consumptions, the lifetime utility of each individual i is then defined as a discounted utility of i and his immediate descendant. As a result, social lifetime utility in the first two periods also corresponds to a discounted utility of the current generation and the next generation. Stationarity further implies that social lifetime utility would be evaluated recursively as a discounted sum of all future utilities in which the discount factor is constant.

We can now state one of our main results. If social preferences are represented by a TSU that satisfies stationarity and respects 2-IPC, then the social lifetime utility is an EDU. Furthermore, social instantaneous utility and the social discount factor are equal to a weighted average of individual instantaneous utilities and a weighted average of individual discount factors, respectively.

Theorem 2. *Assume social preferences to be represented by a TSU lifetime utility U with an instantaneous utility function u and a discount function δ_t. Then, U satisfies* 2-IPC *and* stationarity *if and only if u is a convex combination of $\{u_i\}_{i\in\mathcal{I}}$ and $\delta_t = \delta$ is constant across times, with δ being a convex combination of $\{\delta_i\}_{i\in\mathcal{I}}$.*

Theorem 2 means that to be time consistent, the social lifetime utility must be an EDU func-

[17]Through this recursive relation, all generations of a single family (i.e., a *dynasty*) are linked together by a chain of private intergenerational transfers, countervailing any attempt by the government to redistribute resources across them.

tion; hence, society should respect both stationarity and 2-IPC. In fact, in this situation, the social discount factor can only rest between the minimum and the maximum of individual discount factors, and the social instantaneous utility is a weighted sum of individual instantaneous utilities. Thus, the exact value of the social discount factor and the exact form of the social utility function would depend on the choice of weights. Note that the weights for discount factors can differ from those affecting utilities. This means that society can believe in individual i's judgment about time and place high weight (or even full weight) on her discount factor but be more concerned about individual j's welfare and, consequently, place more weight (or even full weight) on his instantaneous utility. In other words, society can locally arbitrate between a discount factor and individual welfare and generalize this arbitrage across individuals.

Since 2-IPC restricts stream comparisons to streams that only differ in the first two periods, it is conceivable to strengthen this condition and, thus, to remove stationarity. For example, stream restrictions can be relaxed to streams that differ in any two arbitrary successive periods: i.e., for any $t \in \mathbb{N}$, any $x, y, x', y' \in \mathcal{L}$, and any $\mathbf{z} \in \mathcal{L}^\infty$, if $U_{ij}(x_t, y_{t+1}, \mathbf{z}_{-(t,t+1)}) \geq U_{ij}(x'_t, y'_{t+1}, \mathbf{z}_{-(t,t+1)})$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(x_t, y_{t+1}, \mathbf{z}_{-(t,t+1)}) \geq U(x'_t, y'_{t+1}, \mathbf{z}_{-(t,t+1)})$. In view of a recursive evaluation of welfare for every pair of successive generations, a natural question is whether this version of the IPC along with recursive evaluation would imply stationarity of the social lifetime utility U. In order words, in this situation, is stationarity redundant?

The following example proves that stationarity is not useless. Consider a society of 2 individuals $\{1, 2\}$. Suppose that individuals have identical instantaneous utilities but that their discount factors differ, i.e., $\delta_1 \neq \delta_2$. Suppose hence that society has the same instantaneous utility as individuals and adopts the following discount function:

$$d(t) = \frac{1}{t}\delta_1 + (1 - \frac{1}{t})\delta_2.$$

Clearly, this society does not have a constant discounting factor. Therefore, the associated social lifetime utility U violates stationarity. However, it is clear that U satisfies 2-IPC. In fact, with stationarity, the above alternative PC turns is equivalent to 2-IPC.

Now, another possibility to modify 2-IPC is to relax the requirement for the two considered periods to be successive. Namely, altruism would no longer be restricted to only the next generation and rather jumps to a later generation. It can be the case, for instance, that individuals do not care about their children but only about their grandchildren. Is this *postponed altruism* also *perfect* in the sense of time consistency? Surprisingly, as proved below in Proposition 3, the answer is positive. Let us first adapt the IPC to capture the idea of postponed altruism. Fix a $k \in \mathbb{N}$.

Any-2-periods Impartial Pareto Condition (2*-IPC). Let $k, m \in \mathbb{N}$. For any $x, y, x', y' \in \mathcal{L}$ and any $\mathbf{z} \in \mathcal{L}^\infty$, if, for all $ij \in \mathcal{I} \times \mathcal{I}$, $U_{ij}(x_k, y_{k+m}, \mathbf{z}_{-(k,k+m)}) \geq U_{ij}(x'_k, y'_{k+m}, \mathbf{z}_{-(k,k+m)})$, then $U(x_k, y_{k+m}, \mathbf{z}_{-(k,k+m)}) \geq U(x'_k, y'_{k+m}, \mathbf{z}_{-(k,k+m)})$.

This condition, 2*-IPC, requires impartial unanimity to apply only if the compared streams differ for the k-th generation and the $(k+m)$-th generation. Along with stationarity, we can then prove that it also implies a time-consistent society. Furthermore, social lifetime utility and social discount factors are weighted averages of individual utilities and factors.

Proposition 3. *Assume social preferences to be represented by a TSU lifetime utility U with an instantaneous utility function u and a discount function δ_t. Then, U satisfies* 2*-IPC *and* stationarity *if and only if u is a convex combination of $\{u_i\}_{i\in\mathcal{I}}$ and $\delta_t = \delta$ is constant across times, with δ being a convex combination of $\{\delta_i\}_{i\in\mathcal{I}}$.*

Proposition 3 means that if a stationary social lifetime utility evaluates individual welfare such that society is concerned only about the utilities of the current generation and the k-th generation, then the lifetime utility of this society is an EDU. Relative to Theorem 2, where 2-IPC is assumed along with stationarity, Proposition 3 leads to the same utilitarian characterization while assuming 2*-IPC and stationarity. Without delving into the formal proof (featured in the Appendix), to be convinced of this, it is sufficient to consider a situation where individual utilities are identical. Therefore, 2*-IPC implies that the value of the social discount function at time k is a weighted average of $\{\delta_i^{k-1}\}_{i\in\mathcal{I}}$. Since this average is between $(\max_{i\in\mathcal{I}} \delta_i)^{k-1}$ and $(\min_{i\in\mathcal{I}} \delta_i)^{k-1}$, there should exist a $\delta \in [\min_{i\in\mathcal{I}} \delta_i, \max_{i\in\mathcal{I}} \delta_i]$ such that δ^{k-1} corresponds exactly to that weighted average value. Stationarity further implies that this society admits a lifetime utility that has a constant discounting factor, i.e., δ.

Proposition 3 turns out to have surprisingly striking implications. To be time consistent, society only needs to consider the utilities of any two generations that are not necessarily successive. This amounts to the fact that a society affected by a remote generation can be regarded as a society affected by the next generation. This in a sense redresses the prevalence of the belief that a perfect altruistic society cares only about the utility of immediate children and not about distant descendants.[18]

[18]Technically, this result translates the intuition whereby chronological distance becomes meaningless when the time horizon is infinite.

5.2 *Quasi-hyperbolic Social Discounting*

Although time consistency is appealing in economic theory, little of it can be seen in economic policy. This can be either explained by the fact that society lacks the power to commit or by the fact that commitment benefits are overwhelmed by commitment costs. Consequently, a demand for social time consistency must be seen as special rather than universal. The quasi-hyperbolic discounting model of Phelps and Pollak [1968] and Laibson [1997] has long served as a standard norm for economic analysis when time inconsistency arises. We present its representative form.

Definition 2. A lifetime utility $U : \mathcal{L}^\infty \to \mathbb{R}$ admits a *quasi-hyperbolic discounting* form if there exists a continuous function u on $\mathcal{L}$ and parameters $\beta \in (0, 1]$ and $\delta \in (0, 1)$ such that for $z \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = u(z_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(z_t). \tag{5}$$

Of particular interest is the question of the principles that society should respect for social lifetime utility to admit a quasi-hyperbolic discounting form. Such a social lifetime utility being time inconsistent, we already know that it violates stationarity. Hence, a weaker stationarity-like condition is required.

Quasi-stationarity. A lifetime utility U is *quasi-stationary* if, for all $x, y \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$,

$$U(x, \mathbf{z}) \geq U(x, \mathbf{z}') \text{ if and only if } U(x, y, \mathbf{z}) \geq U(x, y, \mathbf{z}').$$

Quasi-stationarity means that the social evaluation of consumption streams relative to the next period is invariant to changes in future periods. Thus, this condition admits the possibility that society could assign to the current consumption a place of relative importance out of proportion to all future consumptions.

It is clear that stationarity implies quasi-stationarity (but not vice versa). Therefore, it is intuitive that quasi-stationarity and 2-IPC are compatible with quasi-hyperbolic social discounting. However, in this situation, β and δ in (5) are indeterminate. From the above analysis, quasi-stationarity and 2-IPC imply the product of β and δ to be a weighted average of individual discount factors. As a result, society can freely choose, for instance, either β or δ to be a weighted average. Such indeterminacy contradicts the democratic intuition that every individual should have a say in every social issue. To avoid this indeterminacy, a stronger condition than 2-IPC is required.

From-first-to-third-period Impartial Pareto Condition (3-IPC). For any $x, y, w, x', y', w' \in \mathcal{L}$

and any $\mathbf{z} \in \mathcal{L}^\infty$, if $U_{ij}(x,y,w,\mathbf{z}) \geq U_{ij}(x',y',w',\mathbf{z})$, for all $ij \in \mathcal{I}\times\mathcal{I}$, then $U(x,y,w,\mathbf{z}) \geq U(x',y',w',\mathbf{z})$.

3-IPC states that if two consumption streams only differ in the first three periods, then the fact that all individuals, true and fictitious, rank the these two streams in the same way would imply that society also adopts this ranking. In this situation, society cares directly about the two next generations, which, in the spirit of Phelps and Pollak [1968], reflects 'imperfect altruism'. However, this imperfectness is not only compatible with quasi-hyperbolic social discounting but also resolves the indeterminacy of β and δ.

Theorem 3. *Assume social preferences to be represented by a TSU lifetime utility U with an instantaneous utility function u and a discount function δ. Then, U satisfies* 3-IPC *and* quasi-stationarity *if and only if there exist nonnegative $\{\alpha_i\}_{i\in\mathcal{I}}$ and $\{\lambda_i\}_{i\in\mathcal{I}}$ with $\sum_i \alpha_i = \sum_i \lambda_i = 1$ such that U admits a quasi-hyperbolic discounting form as defined in (5), with*

$$u = \sum_i \alpha_i u_i \quad \textit{and} \quad \delta = \frac{\sum_i \lambda_i \delta_i^2}{\sum_i \lambda_i \delta_i} \quad \textit{and} \quad \beta = \frac{(\sum_i \lambda_i \delta_i)^2}{\sum_i \lambda_i \delta_i^2}.$$

Furthermore, $\delta \in (\min_{i\in\mathcal{I}} \delta_i, \max_{i\in\mathcal{I}} \delta_i)$ and $\beta \in (\frac{\min_i \delta_i}{\max_i \delta_i}, 1)$.

Theorem 3 shows that if a society respects 3-IPC and quasi-stationarity, then the social lifetime utility has a quasi-hyperbolic discounting form. Social instantaneous utility is a weighted average of individual utility, while the social discount factor δ is a proportion of second-moment to first-moment individual discount factors, and social present bias is a proportion of the square of first-moment to second-moment. This result proves that a deviation from perfect altruism incurs present bias. There is a clear tradeoff between present bias and impatience towards future generations. If society would be less present biased, then it would have to place more weight on a particular individual. As β is close to one, society tends to dictate the discount factor, i.e., $\delta \approx \delta_i$, for some individual i. Moreover, the range of β is determined by the degree of discount factor heterogeneity. If individuals are more diverse in terms of impatience, society would be more present biased. This sheds light on the source of present bias, namely, individual discount factor heterogeneity.

Similar to the discussion in Subsection 3.1, 3-IPC can also be relaxed, allowing the compared streams to be different in three periods that are not necessarily consecutive. Note that these arbitrary three periods have to include the current period to reflect the different arbitrages between current and future generations. This difference is scaled by β.

5.3 *Delayed Social Stationarity*

Social discounting and quasi-hyperbolic social discounting are compatible with 2-IPC and 3-IPC, respectively. This naturally gives rise to the following question: what would social discounting be if we were to force this extension to a more general level, i.e., k-IPC, allowing altruism to extend toward the k next generations? We already observed that 3-IPC triggers dynamic inconsistency. Then, it would not be surprising that k-IPC will also lead to such inconsistency. A deeper issue is that k-IPC may result in more inconsistency as k grows larger. In fact, ignoring the degree of dynamic inconsistency might harm, for example, the sustainability of society. Therefore, inconsistency regulation should be a critical concern for a society in making decisions. In what follows, after the formal definition of k-IPC, we then explore how the intensity of social inconsistency can be characterized through this condition.

k-consecutive-periods Impartial Pareto Condition (k-IPC): For any $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^k$ and any $w \in \mathcal{L}^\infty$, if, $U_{ij}(\mathbf{z}, w) \geq U_{ij}(\mathbf{z}', w)$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}, w) \geq U(\mathbf{z}', w)$.

Consider now the subclass of TSUs satisfying k-IPC along with a stationarity-like condition.

Definition 3. A lifetime utility $U : \mathcal{L}^\infty \to \mathbb{R}$ admits a *k-hyperbolic* form if there exists $0 < \beta_1 \leq \ldots \leq \beta_k \leq 1$ and $\delta \in (0, 1)$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = u(z_1) + \beta_1 \delta u(z_2) + \beta_1 \beta_2 \delta^2 u(z_3) + \cdots + \prod_{\ell=1}^{k} \beta_\ell \sum_{t=\ell+1}^{\infty} \delta^{t-1} u(z_t). \tag{6}$$

This formulation assumes a declining discount factor until period k but a constant discount factor thereafter. The parameters β_ℓ can be thought of as a measure of the 'horizon $(\ell - 1)$' *bias*. Each β_ℓ can also represent the size of the perceived distance between periods $(\ell - 1)$ and ℓ. This definition includes the case of an EDU for $\beta_1 = \cdots = \beta_k = 1$ and the classic quasi-hyperbolic utility for $k = 1$. Note that k-hyperbolic utilities are a subclass of 'semi-hyperbolic' utilities, as proposed in Montiel Olea and Strzalecki [2014], in which $\beta_1, \ldots, \beta_k$ are unrestricted. In contrast, a k-hyperbolic utility requires $\beta_1, \ldots, \beta_k$ to be an increasing sequence, which then translates the existence of a present bias.

The advantage of considering this class of present-biased utilities is at least twofold. First, any present-biased TSU can be approximated by some k-hyperbolic utility. Therefore, replacing TSUs with this class of utilities does not lead to a loss of generality. Second, this parametrized utility is data friendly. One can apply, for instance, MPL (multiple price list) to elicit $\beta_1, \ldots, \beta_k$ and, therefore, fully recover the form of social utility.[19]

k**-Stationarity.** A lifetime utility function U is k*-delayed stationary* if, for all $x \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{c}, \hat{\mathbf{c}} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}_k\mathbf{c}) \geq U(\mathbf{z}_k\hat{\mathbf{c}}) \text{ if and only if } U(x, \mathbf{z}_k\mathbf{c}) \geq U(x, \mathbf{z}_k\hat{\mathbf{c}}).$$

This property, k-delayed stationarity, generalizes classical stationarity, which states that if two consumption streams are identical up to period t, then the ranking between these two streams is preserved after adding the same consumption in the current period and delaying both streams one period further. It is clear that a k-hyperbolic utility satisfies k-stationarity, but it is not true that any utility satisfying k-stationarity has a k-hyperbolic form. Delayed stationarity does not impose any restriction on the rate of impatience before period k. Next, it is natural that when k grows, the stationarity-like property becomes stronger. In other words, if $k > \ell$, then k-stationarity implies ℓ-stationarity. Now, we can formally state our result.

Theorem 4. *A social TSU U is k-stationary and satisfies* $(k+2)$-IPC *if and only if there exist nonnegative numbers α_i and γ_i such that social preferences are represented by a k-hyperbolic social lifetime utility U as in (6), with*

$$u = \sum_i \alpha_i u_i \tag{7}$$

$$\delta = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \tag{8}$$

$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^{\ell}}{\sum_j \gamma_j \delta_j^{\ell-1}} \cdot \frac{\sum_j \gamma_j \delta_j^{k}}{\sum_j \gamma_j \delta_j^{k+1}} \qquad \textit{for all } 1 \leq \ell \leq k. \tag{9}$$

Theorem 4 proves that a society respecting both k-stationarity and $(k+2)$-IPC has a social lifetime utility with a form that is k-hyperbolic. Furthermore, social instantaneous utility is a weighted average of individual utility. Additionally, the social discount factor at horizon ℓ *before* horizon k is a proportion of the weighted average of individual discounting function values at horizon ℓ to that at horizon $(\ell - 1)$. The social discount factor at horizon ℓ *after* horizon k is constant and defined as δ, which is a proportion of the weighted average of individual discounting function values at horizon $(k+1)$ to that at horizon k. Therefore, the social discount factor at horizon $\ell \leq k$ can be decomposed into δ and a period $(\ell - 1)$ bias denoted by β_ℓ.

[19]The empirical elicitation question is beyond the scope of this article. We refer to Montiel Olea and Strzalecki [2014] for MPL and Cohen, Ericson, Laibson, and White [2020] for more general methods.

In fact, Theorem 4 includes Theorem 3 as a special case for $k = 1$. When k goes to infinity, k-stationarity has no bite on stationarity, and k-IPC becomes IPC. Therefore, when $k \to \infty$, Theorem 4 is a special case of Theorem 1, in which each individual lifetime utility is an EDU.

Since a social k-hyperbolic lifetime utility displays decreasing impatience, i.e., present bias, it is natural to explore how the degree of decreasing impatience changes when k increases. Let us first provide a notion of comparative present bias.

Definition 4. A utility U is *more present biased* than utility V if, for any t, s in $\mathbb{N}$ and $x, y, x', y' \in \mathcal{L}$, $U(x, \bar{z}_*) = U(y_t, \bar{z}_{*-t})$, $V(x', \bar{z}_*) = V(y'_t, \bar{z}_{*-t})$, and $V(x'_s, \bar{z}_{*-s}) \leq V(y'_{t+s}, \bar{z}_{*-\{t+s\}})$ implies $U(x_s, \bar{z}_*) \leq U(y_{t+s}, \bar{z}_{*-\{t+s\}})$.

The intuition behind this definition is the following:[20] suppose that one utility U equivalently evaluates two streams, one with consumption x at the current time and the other with further consumption y at t. In contrast, another lifetime utility V ranks similarly, i.e., equivalently evaluates two other streams, the current stream x' and a further stream y' at t. Suppose that all consumption is postponed by the time interval s. Whenever utility V prefers further consumption y' at period $(t+s)$ to closer consumption x' at period s, it is always the case that utility U also prefers further consumption y at $t+s$ than closer consumption x at s. Since U has earlier preference reversal than V, U is said to be more present biased than V.

Proposition 4. *Let there be nonnegative numbers α_i and γ_j such that $\sum_{i\in\mathcal{I}} \alpha_i = \sum_{j\in\mathcal{I}} \gamma_j = 1$. If $k \leq \hat{k}$, then a society characterized by $(\hat{u}, \hat{\delta}, \{\hat{\beta}_\ell\}_{\ell=1}^{\hat{k}})$ defined as in (7,8,9) is more present biased than a society characterized by $(u, \delta, \{\beta_\ell\}_{\ell=1}^{k})$ defined as in (7,8,9).*

This result indicates that when k increases, the social lifetime utility becomes less present biased. Next, we observe that k-stationarity is necessary to characterize a k-hyperbolic social utility. If we replace delayed stationarity with the standard stationarity, then society has to be a dictatorship.

Corollary 1. *Assume that individual preferences and social preferences are represented by EDUs. Social lifetime utility U satisfies k-*IPC *for $k \geq 3$, i.e.,* 3-IPC*, if and only if U is dictatorial.*

As an illustration, consider a society with 2 individuals having identical instantaneous utilities. For simplicity, consider that 3-IPC holds. We know that $u(x) + \delta u(y) + \delta^2 u(z)$ is a convex combination of corresponding individual utilities. Consequently, there exists $\lambda \in [0, 1]$ such that

$$\lambda\delta_1 + (1-\lambda)\delta_2 = \delta \quad \text{and} \quad \lambda\delta_1^2 + (1-\lambda)\delta_2^2 = \delta^2.$$

[20] Prelec [2004] and Quah and Strulovici [2013] suggest different notions of comparative decreasing impatience but based on continuous times.

The only solution must be either $\lambda = 1$ or $\lambda = 0$, which are exactly the two polar cases of dictatorship.

Finally, note that 1-IPC is weaker than 2-IPC. One may expect that this relaxation would provide more room for society to choose its social discount factor. This is actually the case, as shown by the following result.

Corollary 2. *Assume that individual preferences and social preferences are EDUs. Social lifetime utility U satisfies k-*IPC *for $k = 1$, i.e.,* 1-IPC*, if and only if the social instantaneous utility u is a convex combination of individual utility $\{u_i\}_{i \in \mathcal{I}}$.*

This last corollary states that when both individual and social preferences satisfy time consistency, if society obeys 1-IPC, then the social instantaneous utility has to be weighted utilitarian. This condition, 1-IPC, imposes no restriction on the social choice of the discount factor. On the one hand, one may believe a society is benevolent and that 1-IPC is helpful to make a better choice of discount factor, which goes beyond the individualism restriction. On the other hand, endowing enormous latitude without individual approvals might incur more conflicts and instability.

6 Conclusion

Chambers and Echenique [2018] and Weitzman [2001] illustrate the undeniable and substantial reality of individual heterogeneity, that is, the natural disagreement between individuals' feelings or experts' opinions over lifetime preferences, which should encapsulate the tradeoff between current benefits and future benefits from a social perspective. Indeed, it is well known from Harsanyi [1955] and Zuber [2011] that a society respecting the Pareto condition (PC) cannot make decisions in the same way as individuals when they are heterogeneous in lifetime preferences, whether it comes from the heterogeneity of discount factors, instantaneous utilities or both. Our paper demonstrates, however, that a society satisfying another kind of Paretian unanimity, namely, the so-called impartial Pareto condition (IPC), behaves in the same way as individuals when individuals' preferences are time separable. In addition, the social discount function and instantaneous utility are shown to correspond to a weighted average of individual discount functions and individual utilities, respectively.

The starting objective of our paper was to identify both normative and descriptive mechanisms to aggregate heterogeneous time preferences when individuals are constant discounters. It appears, first, that the appropriate normative mechanism ensuring society to behave as a constant discounter and the social discount factor to be a weighted mean of individual discount factors requires society to obey 2-IPC, which is on the one hand a weaker postulate than the PC and, on the other, a postulate avoiding imperfect altruism in the sense of Phelps and Pollak [1968], which is known to generate violations of stationarity. Second, a popular descriptive mechanism comprehends society as a quasi-hyperbolic discounter. We show that this mechanism requires society to obey, in this case, 3-IPC. In addition, the associated aggregation rule is such that the parameters β and δ, as defined in (5), are uniquely determined by individual discount factors. Finally, this descriptive mechanism can be generalized from quasi-hyperbolic discounting to any arbitrary k-hyperbolic discounting. In this way, different rates of impatience exhibiting different rates of bias can be considered comparatively.

APPENDIX

A PRELIMINARY RESULTS

We first prove a general aggregation result, worded here as Proposition A1. One may find different versions of this proof, but we choose to present this result for two reasons. On the one hand, it is expressed in our setting, and on the other hand, it will be used repeatedly in the following proofs. To avoid tiresome duplication, this result is then singled out at the beginning.

If $k \in \mathbb{N}$, for $i, j \in \mathcal{I} \times \mathcal{I}$, let $U_{ij}^k : \mathcal{L}^k \to \mathbb{R}$ and $U^k : \mathcal{L}^k \to \mathbb{R}$ be two real-valued functions defined on $\mathcal{L}^k$ with a convex range. Consider now a unanimity postulate.

k-Unanimity: Fix $k \in \mathbb{N}$; for any $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^k$, if $U_{ij}^k(\mathbf{z}) \geq U_{ij}^k(\mathbf{z}')$, for all $i, j \in \mathcal{I} \times \mathcal{I}$, then $U^k(\mathbf{z}) \geq U^k(\mathbf{z}')$. Furthermore, if there exists a fictitious ij such that $U_{ij}^k(\mathbf{z}) > U_{ij}^k(\mathbf{z}')$, then $U^k(\mathbf{z}) > U^k(\mathbf{z}')$.

Proposition A1. *Under MAC,*[21] *k-unanimity holds if and only if there exist positive numbers λ_{ij} and a real number μ such that, for $\mathbf{z} \in \mathcal{L}^k$:*

$$U^k(\mathbf{z}) = \sum_{ij \in \mathcal{I} \times \mathcal{I}} \lambda_{ij} U_{ij}^k(\mathbf{z}) + \mu.$$

Proof. Let

$$Y = \{\mathbf{y} \in \mathbb{R}^{n^2+1} | y_0 \leq -1 \text{ and } y_{ij} \geq 0, \text{ for } ij \in \mathcal{I} \times \mathcal{I}\},$$

[21]Recall the *minimum agreement condition* (MAC): There exists $z^*, z_* \in \mathcal{L}$ such that, for all $k \in \mathbb{N}$ and all $ij \in \mathcal{I} \times \mathcal{I}$, $U_{ij}(z^*) > U_{ij}(z_*)$.

and

$$A = \Big\{ \big(U^k(\mathbf{z}) - U^k(\hat{\mathbf{z}}), U_{11}^k(\mathbf{z}) - U_{11}^k(\hat{\mathbf{z}}), \dots, U_{nn}^k(\mathbf{z}) - U_{nn}^k(\hat{\mathbf{z}}) \big) \in \mathbb{R}^{n^2+1} \big| \mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^k \Big\}.$$

Since $\mathcal{L}$ is compact, the set $\{(U^k(\mathbf{z}), U_{11}^k(\mathbf{z}), \dots, U_{nn}^k(\mathbf{z})) \big| \mathbf{z} \in \mathcal{L}\}$ is convex. Therefore, A is convex and symmetric with respect to vector $\overrightarrow{0}$. According to k-unanimity, we have $Y \cap A = \emptyset$. Now, define the vector space spanned by A:

$$\mathrm{span}(A) = \Big\{ \sum_{\ell=1}^{m} r_\ell \mathbf{a}_\ell \big| m \in \mathbb{N}, r_\ell \in \mathbb{R} \text{ and } \mathbf{a}_\ell \in A \Big\}.$$

It is immediately clear that $Y \cap \mathrm{span}(A) = \emptyset$. Since Y and $\mathrm{span}(A)$ are polyhedral, nonempty and mutually disjoint, the strictly separating theorem (see, e.g., Rockafellar, Corollary 19.3.3) means that there exist $\boldsymbol{\pi} = (\pi, \pi_{11}, \dots, \pi_{nn})$ such that, for all $\mathbf{a} \in \mathrm{span}(A)$ and $\mathbf{y} \in Y$, $\boldsymbol{\pi} \cdot \mathbf{y} > \boldsymbol{\pi} \cdot \mathbf{a}$. Note that for all $\mathbf{a} \in \mathrm{span}(A)$, we have $\boldsymbol{\pi} \cdot \mathbf{a} = 0$. (Suppose the opposite. Then, there must exist $\hat{\mathbf{a}} \in \mathrm{span}(A)$ such that $\boldsymbol{\pi} \cdot \hat{\mathbf{a}} > 0$, and this wlog since set A is symmetric. Therefore, there exists a large enough $r \in \mathbb{R}$ such that $\boldsymbol{\pi} \cdot r\hat{\mathbf{a}} > \boldsymbol{\pi} \cdot \mathbf{y}$, which is a contradiction.)

Select $\mathbf{y} = (-1, 0, \dots, 0) \in Y$. The above inequality, i.e., $\boldsymbol{\pi} \cdot \mathbf{y} > 0$, implies that $-\pi > 0$, that is, $\pi < 0$. Thus, for all $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^k$,

$$U^k(\mathbf{z}) - U^k(\hat{\mathbf{z}}) = \sum_{ij \in \mathcal{I} \times \mathcal{I}} \frac{\pi_{ij}}{-\pi} \big[U_{ij}^k(\mathbf{z}) - U_{ij}^k(\hat{\mathbf{z}}) \big].$$

Fix $\hat{\mathbf{z}}$. For $ij \in \mathcal{I} \times \mathcal{I}$, define λ_{ij} as $\frac{\pi_{ij}}{-\pi}$. Define also μ as $U^k(\hat{\mathbf{z}}) - \sum_{ij \in \mathcal{I} \times \mathcal{I}} \lambda_{ij} U_{ij}^k(\hat{\mathbf{z}})$. Therefore, for all $\mathbf{z} \in \mathcal{L}^k$, we have $U^k(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}^k(\mathbf{z})$.

To verify that each λ_{ij} is positive, let $\mathbf{y}$ be such that $y = -1, y_{ij} = r > 0$ and $y_{i'j'} = 0$ for $ij \neq i'j'$. The existence of $\mathbf{y}$ is guaranteed by MAC. Therefore, $\boldsymbol{\pi} \cdot \mathbf{y} > 0$ implies $-\pi + r\pi_{ij} > 0$. Hence, $\pi_{ij} > 0$, for ij, which then entails that each $\lambda_{ij} > 0$. □

B PROOF OF SECTION 3: PROPOSITION 1

The necessity part whereby a dictatorial society satisfies the PC is straightforward.
Sufficiency part. Note that if we assume that $k = \infty$ and $U_{ij}(\mathbf{z}) = \sum_t d_{it} u_i(z_t)$, for all $j \in \mathcal{I}$, then the PC is equivalent to k-unanimity, for $k = \infty$. Since the PC is satisfied, by Proposition A1, for

$k = \infty$, there exist nonnegative $\{\lambda_i\}_{i\in\mathcal{I}}$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$:

$$\sum_t d_t u(z_t) = \sum_{i\in\mathcal{I}} \lambda_i \sum_t d_{it} u_i(z_t) + \mu.$$

By normalization, $u_i(z_*) = 0$ and $u_i(z^*) = 1$ for all i. First, take $\mathbf{z} = (z_*, \ldots, z_*)$. Then, we have $\mu = 0$. Second, take $\mathbf{z} = (z^*, z_*, z_*, \ldots)$. Then, it implies $\sum_{i\in\mathcal{I}} \lambda_i = 1$. Therefore, for all $t \in \mathbb{N}$, $d_t = \sum_{i\in\mathcal{I}} \lambda_i d_{it}$, and for all $z \in \mathcal{L}$, $u(z) = \sum_{i\in\mathcal{I}} \lambda_i u_i(z)$. Hence, it requires that $\sum \lambda_i d_{it} u_i(z) = \sum \lambda_i d_{it} \cdot \sum \lambda_i u_i(z)$, i.e. :

$$\sum_{i\in\mathcal{I}} \lambda_i \big(d_{it} - \sum_{i\in\mathcal{I}} \lambda_i d_{it}\big) u_i(z) = 0.$$

By regularity, we know that there exists a subset $J \subseteq \mathcal{I}$ such that all $\{u_i\}_{i\in J}$ are independent. Thus, for every $i \in J$, $d_{it} - \sum \lambda_i d_{it} = 0$. Hence, $\lambda_i = 0$ or 1.

C Proofs of Section 4

C.1 *Proof of Theorem 1*

Necessity part. Suppose that for $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, all $ij \in \mathcal{I}\times\mathcal{I}$, $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$. Since each α_i and γ_j are nonnegative, then we have $\alpha_i\gamma_j U_{ij}(\mathbf{z}) \geq \alpha_i\gamma_j U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I}\times\mathcal{I}$. Now, $U = \sum_{ij} \alpha_i\gamma_j U_{ij}$ implies $U(\mathbf{z}) \geq U(\mathbf{z}')$, which proves the IPC.

Sufficiency part. Suppose that the IPC is satisfied. The IPC is equivalent to k-unanimity for $k = \infty$, where $U_{ij}(\mathbf{z}) = \sum_t d_{jt} u_i(z_t)$, for all $\mathbf{z} \in \mathcal{L}^\infty$. Then, according to Proposition A1, there exist nonnegative $\{\lambda_{ij}\}_{ij\in\mathcal{I}\times\mathcal{I}}$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$, $U(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}(\mathbf{z})$, i.e.:

$$\sum_{t=1}^{\infty} d_t u(z_t) = \sum_{ij} \lambda_{ij} \sum_{t=1}^{\infty} d_{jt} u_i(z_t).$$

Recall that, by normalization, $u_i(z_*) = 0$ and $u_i(z^*) = 1$ for all i. Accordingly, defining $\mathbf{z}$ as $z_1 = z^*$ and $z_t = z_*$, for $t \neq 1$, implies $\sum_{ij} \lambda_{ij} = 1$. Let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, for $z \in \mathcal{L}$, $u(z) = \alpha_i u_i(z)$. For $t \in \mathbb{N}$, consider the stream $\mathbf{z}$ such that $z_t = z^*$ and $z_s = z_*$, for $s \neq t$. Therefore, $d_t = \sum_j \gamma_j d_{jt}$. Hence, U separately aggregates instantaneous individual utilities and individual discount functions.

C.2 Proof of Lemma 1

Necessity part. Suppose that $U(x_t, \bar{z}_{*-t}) = U(y_s, \bar{z}_{*-s})$, where $t > s$. This implies, for $k > 0$:

$$\begin{aligned}
\frac{u(y)}{u(x)} = \frac{d_t}{d_s} &= \frac{d_t}{d_{t-1}} \times \frac{d_{t-1}}{d_{t-2}} \times \cdots \times \frac{d_{s+1}}{d_s} \\
&= \delta(t) \times \delta(t-1) \times \cdots \times \delta(s+1) \\
&\leq \delta(t+k) \times \delta(t+k-1) \times \cdots \times \delta(s+k+1) \\
&= \frac{d_{t+k}}{d_{s+k}}.
\end{aligned}$$

Therefore, it is clear that $U(x_{t+k}, \bar{z}_{*-(t+k)}) \geq U(y_{s+k}, \bar{z}_{*-(s+k)})$.

Sufficiency part. Since u is continuous and has a range lying between 0 and 1, for $t > 1$, there exist $x, y \in \mathcal{L}$ such that:

$$\delta(t) = \frac{d_t}{d_{t-1}} = \frac{u(x)}{u(y)}.$$

Then, present bias implies, for $k \in \mathbb{N}$, $d_{t+k}u(x) \geq d_{t+k-1}u(y)$. Therefore, we have:

$$\frac{d_t}{d_{t-1}} \geq \frac{d_t}{d_{t-1}}.$$

Since this expression is valid for all t, the discount factor $\delta(t)$ is increasing in t.

A similar process can be easily applied to prove the case of constant impatience.

C.3 Proof of Proposition 2

By Lemma 1, it suffices to show that $\delta(t)$ is increasing. Let $\delta_i(t)$ be the discount factor of individual i at horizon t. Then,

$$\begin{aligned}
\delta(t+1) = \frac{d_{t+1}}{d_t} &= \frac{\sum \gamma_i d_{i(t+1)}}{\sum \gamma_i d_{it}} \\
&= \frac{\sum \gamma_i \delta_i(t+1) d_{it}}{\sum \gamma_i d_{it}}.
\end{aligned}$$

Similarly, we have:

$$\delta(t) = \frac{\sum \gamma_i d_{it}}{\sum \frac{\gamma_i}{\delta_i(t)} d_{it}}.$$

Since $\delta_i(t)$ is increasing, for all i, it follows therefrom:

$$\frac{\delta_j(t+1)}{\delta_i(t)} + \frac{\delta_i(t+1)}{\delta_j(t)} \geq 2\sqrt{\frac{\delta_j(t+1)}{\delta_i(t)} \times \frac{\delta_i(t+1)}{\delta_j(t)}} \geq 2.$$

Now, since coefficients γ_i are nonnegative, it is also true that:

$$\Big(\frac{\delta_j(t+1)}{\delta_i(t)} + \frac{\delta_i(t+1)}{\delta_j(t)}\Big)\gamma_i\gamma_j \geq 2\gamma_i\gamma_j \quad \text{for all } i,j \in \mathcal{I}.$$

Therefore:

$$\Big(\sum \gamma_i\delta_i(t+1)d_{it}\Big) \times \Big(\sum \frac{\gamma_i}{\delta_i(t)}d_{it}\Big) \geq \Big(\sum \gamma_i d_{it}\Big)^2,$$

which, in turn, implies that $\delta(t+1) \geq \delta(t)$.

D Proofs of Section 5

D.1 Proof of Theorem 2

Necessity part is straightforward.

Sufficiency part. Suppose that 2-IPC and stationarity hold.

(i) We have to show that the social lifetime utility U is an EDU. For this, it is sufficient to demonstrate that U satisfies all postulates of Koopmans [1960]. First, Postulate 1 is implied by the continuity of U. Next, Postulates 3 and 3′ are implied by the time separability of U. Moreover, stationarity corresponds to Postulate 4. Since $u_i(z^*) > u_i(z_*)$, for all $i \in \mathcal{I}$, 2-IPC implies that $u(z^*) > u(z_*)$. Therefore, by time additivity, for $\mathbf{z} \in \mathcal{L}^\infty$, $U(z^*, \mathbf{z}) > U(z_*, \mathbf{z})$, which implies Postulate 2. Finally, since $u(z^*) \geq u(z) \geq u(z_*)$, for all $z \in \mathcal{L}$ and all $\mathbf{z} \in \mathcal{L}^\infty$, we have $U(z^*, \ldots, z^*) \geq U(\mathbf{z}) \geq U(z_*, \ldots, z_*)$, that is, Postulate 5. Hence, by Koopmans' Theorem, the social lifetime utility U is an EDU.

(ii) For $k = 2$, 2-IPC is equivalent to 2-unanimity, where $U^2_{ij}(x,y) = u_i(x) + \delta_j u_i(y)$ and $U^2(x,y) = u(x) + \delta u(y)$, for all $(x,y) \in \mathcal{L}^2$. Therefore, Proposition A1 implies that there exist nonnegative λ_{ij} and a real number μ such that $U^2 = \lambda_{ij}U^2_{ij} + \mu$, that is, for any $(x,y) \in \mathcal{L}^2$:

$$u(x) + \delta u(y) = \sum_{ij} \lambda_{ij}u_i(x) + \sum_{ij} \lambda_{ij}\delta_j u_i(y) + \mu.$$

Let $x = y = z_*$. Then, $u(z_*) = u_i(z_*) = 0$ implies $\mu = 0$. Now, take $y = z_*$, and let $\alpha_i = \sum_j \lambda_{ij}$. The above equation becomes $u(x) = \sum_i \alpha_i u_i(x)$,, which proves that the social instantaneous utility u is a convex combination of individual instantaneous utilities. Suppose that $x = z_*$ and $y = z^*$. Then, $u(z^*) = u_i(z^*) = 1$, for all i. Let $\gamma_j = \sum_i \lambda_{ij}$. The above equation then becomes $\delta = \sum_j \gamma_j \delta_j$,, which proves that the social discount factor δ is a convex combination of individual discount factors.

D.2 Proof of Proposition 3

Necessity part. Stationarity is immediate. We have only to prove 2^*-IPC. Fix $k, m \in \mathbb{N}$. Suppose that $U_{ij}(x_k, y_{k+m}, \mathbf{z}_{-(k,k+m)}) \geq U_{ij}(x'_k, y'_{k+m}, \mathbf{z}_{-(k,k+m)})$, for each $ij \in \mathcal{I} \times \mathcal{I}$. Since U is an EDU, this is equivalent to $u_i(x) + \delta_j^m u_i(y) \geq u_i(x') + \delta_j^m u_i(y')$. We know that $u = \sum_{i \in \mathcal{I}} \alpha_i u_i$ and $\delta = \sum_{j \in \mathcal{I}} \gamma_j \delta_j$, both α_i and γ_j being nonnegative with $\sum_i \alpha_i = \sum_j \gamma_j = 1$. Therefore, there exist nonnegative $\hat{\gamma}_j$ with $\sum_j \hat{\gamma}_j = 1$ such that $\delta^m = \sum_j \hat{\gamma}_j \delta_j^m$. Hence,

$$\sum_i \hat{\gamma}_j \sum_i \alpha_i \big(u_i(x) + \delta_j^m u_i(y)\big) \geq \sum_i \hat{\gamma}_j \sum_i \alpha_i \big(u_i(x') + \delta_j^m u_i(y')\big),$$

i.e., $u(x) + \delta^m u(y) \geq u(x') + \delta^m u(y')$. Thus, we have:

$$U(x_k, y_{k+m}, \mathbf{z}_{-(k,k+m)}) \geq U(x'_k, y'_{k+m}, \mathbf{z}_{-(k,k+m)}).$$

Sufficiency part. Suppose that stationarity and 2^*-IPC hold. By a similar argument as that used in the proof of Theorem 2, we know that the social lifetime utility U is an EDU (u, δ). Therefore, 2^*-IPC is equivalent to k-unanimity for $k = 2$ and $U_{ij}^2 = u_i + \delta_j^m u_i$. According to Proposition A1, there exist nonnegative λ_{ij} and μ such that, for $x, y \in \mathcal{L}$, $u(x) + \delta^m u(y) = \sum_{ij} \lambda_{ij} \big(u_i(x) + \delta_j^m u_i(y)\big) + \mu$. Furthermore, $\mu = 0$ and $\sum_{ij} \lambda_{ij} = 1$. Moreover, let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. We know that $u = \sum_i \alpha_i u_i$ and $\delta^m = \sum_j \gamma_j \delta_j^m$. Furthermore, since $\delta^m \in [\min_j \delta_j^m, \max_j \delta_j^m]$, it means that $\delta \in [\min_j \delta_j, \max_j \delta_j]$. Hence, there exist nonnegative $\hat{\gamma}_j$, with $\sum_j \hat{\gamma}_j = 1$, such that $\delta = \sum_j \hat{\gamma}_j \delta_j$.

D.3 Proof of Theorem 4

If $k \in \mathbb{N}$, we define the function $U^k : \mathcal{L}^\infty \to \mathbb{R}$ for $\mathbf{z} \in \mathcal{L}^\infty$ as follows:

$$U^k(\mathbf{z}) = U(z_*, \ldots, z_*, \mathbf{z}) = \sum_{t=1}^{\infty} d_{t+k} u(z_t).$$

We want first to show that this so-defined function U^k satisfies all five postulates of Koopmans [1960] and, therefore, is an EDU.

(i) Postulate 1 follows from the definition of U^k and continuity of u.

(ii) Since $u_i(z^*) > u_i(z_*)$, for all $i \in \mathcal{I}$, k-unanimity implies that $u(z^*) > u(z_*)$. Therefore, due to the time additivity of U^k and $d_{k+1} > 0$, for $\mathbf{z} \in \mathcal{L}^\infty$, $U^k(z^*, \mathbf{z}) > U^k(z_*, \mathbf{z})$, which implies Postulate 2.

(iii) Postulate 3 follows immediately from the time additivity of U^k. That is, for all $x, y \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$:

$$U^k(x, \mathbf{z}) \geq U^k(y, \mathbf{z}) \Leftrightarrow u(x) \geq u(y) \Leftrightarrow U^k(x, \mathbf{z}') \geq U^k(y, \mathbf{z}').$$

Similarly, we have:

$$U^k(x, \mathbf{z}) \geq U^k(x, \mathbf{z}') \Leftrightarrow \sum_{t=2}^{\infty} u(z_t) \geq \sum_{t=2}^{\infty} u(z'_t) \Leftrightarrow U^k(y, \mathbf{z}) \geq U^k(y, \mathbf{z}').$$

(iv) Let $x \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, such that:

$$\begin{aligned} U^k(\mathbf{z}) \geq U^k(\mathbf{z}') &\Leftrightarrow U(z_*, \ldots, z_*, \mathbf{z}) \geq U(z_*, \ldots, z_*, \mathbf{z}') \\ &\Leftrightarrow U(z_*, \ldots, x, \mathbf{z}) \geq U(z_*, \ldots, x, \mathbf{z}') \\ &\Leftrightarrow U(z_*, \ldots, x, \mathbf{z}) \geq U(z_*, \ldots, x, \mathbf{z}') \\ &\Leftrightarrow U^k(x, \mathbf{z}) \geq U^k(x, \mathbf{z}'). \end{aligned}$$

The first and last equivalence relations are given by definition. The second equivalence stems from the time additivity of U. The third equivalence is induced by the property of k-stationarity and proves that Postulate 4 holds.

(v) Note that $u(z^*) > u(z_*)$. Therefore, for two streams $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$ such that $z_t = z^*$ and $z'_t = z_*$, for all $t \in \mathbb{N}$, because d_t is positive, for any $\hat{\mathbf{z}} \in \mathcal{L}^\infty$, we have:

$$U^k(\mathbf{z}) \geq U^k(\hat{\mathbf{z}}) \geq U^k(\mathbf{z}'),$$

i.e., Postulate 5.

(vi) Finally, we have to demonstrate Postulate $3'$. Let $x, y, x', y' \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$. Hence:

$$\begin{aligned} U^k(x, y, \mathbf{z}) \geq U^k(x', y', \mathbf{z}) &\Leftrightarrow d_{k+1}u(x) + d_{k+2}u(y) \geq d_{k+1}u(x') + d_{k+2}u(y') \\ &\Leftrightarrow U^k(x, y, \mathbf{z}') \geq U^k(x', y', \mathbf{z}'). \end{aligned}$$

Similarly:

$$U^k(x,y,\mathbf{z}) \geq U^k(x',y,\mathbf{z}') \Leftrightarrow d_{k+1}u(x) + \sum_{t=2}^{\infty} d_{t+2}u(z_t) \geq d_{k+1}u(x') + \sum_{t=2}^{\infty} d_{t+2}u(z'_t)$$
$$\Leftrightarrow U^k(x,y',\mathbf{z}) \geq U^k(x',y',\mathbf{z}').$$

Therefore, U^k defined on $\mathcal{L}^\infty$ satisfies Postulates 1-5 and $3'$. Then, according to Koopmans' Theorem, there exist $a \in \ ]0,1[$ and a continuous function u on $\mathcal{L}$, such that:

$$U^k(\mathbf{z}) = \sum_{t=1}^{\infty} a^{t-1}u(z_t).$$

Since the representation is unique, there exists $b > 0$ such that, for $\mathbf{z} \in \mathcal{L}^\infty$:

$$U(\mathbf{z}) = \sum_{t=1}^{k} d_t u(z_t) + b \sum_{t=k+1}^{\infty} a^{t-k-1}u(z_t).$$

Wlog, we can normalize; i.e., $d_1 = 1$, $u(z_*) = 0$ and $u(z^*) = 1$. For $\mathbf{z} \in \mathcal{L}^\infty$, we write:

$$U^{k+2}(\mathbf{z}) = \sum_{t=1}^{k} d_t u(z_t) + bu(z_{k+1}) + bau(z_{k+2}).$$

Therefore, $(k+2)$-unanimity can be equivalently written as follows: for any $z_1, \ldots, z_{k+2}$ and $z'_1, \ldots, z'_{k+2}$ in $\mathcal{L}$,

$$U^{k+2}_{ij}(\mathbf{z}) \geq U^{k+2}_{ij}(\mathbf{z}'), \text{ for all } i,j \in \mathcal{I} \Longrightarrow U^{k+2}(\mathbf{z}) \geq U^{k+2}(\mathbf{z}').$$

Hence, there exist nonnegative λ_{ij} such that:

$$U^{k+2}(\mathbf{z}) = \sum_{ij} \lambda_{ij} U^{k+2}_{ij}(\mathbf{z}). \tag{10}$$

Let $\mathbf{z}$ be such that $z_1 = z^*$ and $z_t = z_*$ for $t \neq 1$. Then, (10) implies $\sum_{ij} \lambda_{ij} = 1$. For $i \in \mathcal{I}$, denote $\alpha_i = \sum_j \lambda_{ij}$. Therefore, for all $z \in \mathcal{L}$, $u(z) = \sum_i \alpha_i u_i(z)$, which proves that u is a convex combination of u_i.

For $j \in \mathcal{I}$, denote $\gamma_j = \sum_i \lambda_{ij}$. Clearly, $\sum_j \gamma_j = 1$. Now, let $\mathbf{z}, \mathbf{z}'$ be such that $z_{k+1} = z^*$ and $z_t = z_*$, for $z \neq k+1$, $z'_{k+2} = z^*$ and $z_t = z_*$, for $z \neq k+2$. Substituting $\mathbf{z}$ and $\mathbf{z}'$ into (10) implies $b = \sum_j \gamma_j \delta_j^k$ and $ba = \sum_j \gamma_j \delta_j^{k+1}$. Define $\delta = a$ as follows:

$$\delta := a = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k}. \tag{11}$$

Let $\overline{\delta} = \max_j \delta_j$ and $\underline{\delta} = \min_j \delta_j$. Therefore, since γ_j is nonnegative, for any j:

$$\frac{\sum_j \gamma_j \delta_j^k \underline{\delta}}{\sum_j \gamma_j \delta_j^k} \leq \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \leq \frac{\sum_j \gamma_j \delta_j^k \overline{\delta}}{\sum_j \gamma_j \delta_j^k},$$

i.e., $\underline{\delta} \leq \delta \leq \overline{\delta}$.
Let $\hat{b}$ be such that $\hat{b} \cdot a^k = b$. Therefore:

$$\hat{b} = \frac{(\sum_j \gamma_j \delta_j^k)^{k+1}}{(\sum_j \gamma_j \delta_j^{k+1})^k}.$$

Let $\mathbf{z}$ be such that $z_k = z^*$ and $z_t = z_*$ for $t \neq k$. Substituting $\mathbf{z}$ into (10) implies $d_k = \sum_j \gamma_j \delta_j^{k-1}$. Similarly, for all $\ell = 2, \ldots, k-1$, $d_\ell = \sum_j \gamma_j \delta_j^{\ell-1}$. Now, define $\beta_k, \ldots, \beta_1$ recursively:

$$\begin{aligned}
\beta_k &= \hat{b} \times \frac{\delta^{k-1}}{d_k} \\
\beta_{k-1} &= \frac{\hat{b}}{\beta_k} \times \frac{\delta^{k-2}}{d_{k-1}} \\
&\vdots \\
\beta_\ell &= \frac{\hat{b}}{\beta_k \beta_{k-1} \cdots \beta_{\ell+1}} \times \frac{\delta^{\ell-1}}{d_\ell} \\
&\vdots \\
\beta_1 &= \frac{\hat{b}}{\beta_k \beta_{k-1} \cdots \beta_2}.
\end{aligned}$$

Hence, for every $\ell = 2, \ldots, k$, it yields $d_\ell = \beta_1 \beta_2 \cdots \beta_{\ell-1} \delta^{\ell-1}$. Thus, for $\mathbf{z} \in \mathcal{L}^\infty$, U should take the following form:

$$U(\mathbf{z}) = u(z_1) + \beta_1 \delta u(z_2) + \beta_1 \beta_2 \delta^2 u(z_3) + \cdots + \prod_{j=1}^{k} \beta_j \sum_{t=j+1}^{\infty} \delta^{t-1} u(z_t),$$

in which δ is given by (11). Furthermore, substituting $\hat{b}$ and each d_ℓ into the above expression implies:

$$\beta_k = \frac{(\sum_j \gamma_j \delta_j^k)^2}{(\sum_j \gamma_j \delta_j^{k-1})(\sum_j \gamma_j \delta_j^{k+1})}$$
$$\beta_{k-1} = \frac{\sum_j \gamma_j \delta_j^{k-1}}{\sum_j \lambda_j \delta_j^{k-2}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}$$
$$\vdots$$
$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}$$
$$\vdots$$
$$\beta_1 = \frac{\sum_j \gamma_j \delta_j}{\sum_j \gamma_j} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}.$$

Now, we need to show that $0 < \beta_\ell < 1$, for each $\ell = 1, \ldots, k$. Let $1 < \ell < k$. Denote by A the term $\left[\sum_j \gamma_j \delta_j^\ell)^2 - (\sum_j \gamma_j \delta_j^{\ell-1})(\sum_j \gamma_j \delta_j^{\ell+1}\right]$. Then, we have:

$$\begin{aligned} A &= \sum_j (\gamma_j \delta_j^\ell)^2 + 2\sum_{i<j} \gamma_i \gamma_j \delta_i^\ell \delta_j^\ell - \big(\sum_j (\gamma_j \delta_j)^2 + \sum_{i<j} \gamma_i \gamma_j \delta_i^{\ell-1} \delta_j^{\ell+1} + \sum_{i<j} \gamma_i \gamma_j \delta_i^{\ell+1} \delta_j^{\ell-1}\big) \\ &= \sum_{i<j} \gamma_i \gamma_j (\delta_i \delta_j)^2 \big(2\delta_i \delta_j - \delta_i^2 - \delta_j^2\big) = -\sum_{i<j} \gamma_i \gamma_j (\delta_i \delta_j)^2 \big(\delta_i - \delta_j^2\big)^2 < 0. \end{aligned}$$

Since $A < 0$, it implies:

$$\frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \le \frac{\sum_j \gamma_j \delta_j^{\ell+1}}{\sum_j \gamma_j \delta_j^\ell}.$$

By induction, we obtain:

$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}} \le \frac{\sum_j \gamma_j \delta_j^{k+1}}{(\sum_j \gamma_j \delta_j)^k} \times \frac{\sum_j \gamma_j \delta_j^k}{(\sum_j \gamma_j \delta_j)^{k+1}} = 1.$$

D.4 Proof of Theorem 3

Note that Theorem 3 is a special case of Theorem 4 in which $k = 1$. Therefore, its proof follows directly from the proof of Theorem 4.

D.5 *Proof of Proposition 4*

Let $k < k'$. Let α_i and γ_j be nonnegative numbers such that $\sum_{i\in\mathcal{I}} \alpha_i = \sum_{j\in\mathcal{I}} \gamma_j = 1$. Therefore, according to (7,8,9):

$$\begin{aligned} u = \sum_i \alpha_i u_i \quad &= \quad \hat{u} \\ \delta = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^{k}} \quad &< \quad \hat{\delta} = \frac{\sum_j \gamma_j \delta_j^{\hat{k}+1}}{\sum_j \gamma_j \delta_j^{\hat{k}}} \\ \beta_\ell = \frac{\sum_j \gamma_j \delta_j^{\ell}}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^{k}}{\sum_j \gamma_j \delta_j^{k+1}} \quad &> \quad \hat{\beta}_\ell = \frac{\sum_j \gamma_j \delta_j^{\ell}}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^{\hat{k}}}{\sum_j \gamma_j \delta_j^{\hat{k}+1}}. \end{aligned}$$

Let $t, s \in \mathbb{N}$ with $t > 1$. Consider consumptions $x, y, x', y' \in \mathcal{L}$, such that $U(x, \bar{z}_*) = U(y_t, \bar{z}_{*-t})$, $\hat{U}(x', \bar{z}_*) = \hat{U}(y'_t, \bar{z}_{*-t})$, and $U(x_s, \bar{z}_{*-s}) \leq U(y_{t+s}, \bar{z}_{*-(t+s)})$. Equivalently, we have:

$$\begin{aligned} u(x) \quad &= \quad d_t u(y) \\ \hat{u}(x') \quad &= \quad \hat{d}_t \hat{u}(y') \\ d_s u(x) \quad &\leq \quad d_{t+s} u(y). \end{aligned}$$

Therefore,

$$d_t d_s \leq d_{t+s}. \tag{12}$$

We need to show that $\hat{d}_t \hat{d}_s \leq \hat{d}_{t+s}$. Hence, consider the three following cases:

<u>**Case 1:**</u> $k \geq t + s$.

Then, (12) implies:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_{t+s}} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_{t+s}} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}}.$$

Since $\delta < \hat{\delta}$, it is straightforward that:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} < \hat{\delta},$$

which implies: $\hat{d}_t \hat{d}_s \leq \hat{d}_{t+s}$.

Case 2: $k < t + s \leq \hat{k}$.

Assume that $t \leq k$ and $s \leq k$. (For the case of $t \geq k$ or $s \geq k$, the proof is quite similar.) Then, (12) implies:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} \times \left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s}.$$

Note that, for $1 \leq \ell \leq t + s - k$:

$$\left(\frac{\hat{\delta}}{\delta}\right) \times \hat{\beta}_{k+\ell} \geq 1,$$

which implies:

$$\left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s} \geq 1.$$

Therefore:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} < \hat{\delta}.$$

Case 3: $t + s > \hat{k}$.

We only prove the case corresponding to $t \leq k$ and $s \leq k$ since the rest are similar. Again, we have:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} < \delta.$$

In this case, we know that:

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{\hat{k}}} \times \left(\frac{\hat{\delta}}{\delta}\right)^{t+s-\hat{k}} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{\hat{k}}.$$

By the same argument as the one used in Case 2, we have:

$$\left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s} \geq 1.$$

Therefore:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{\hat{k}}} < \hat{\delta}.$$

D.6 Proof of Corollary 1

The necessity part is immediate.

Sufficiency part. Suppose the social lifetime utility U satisfies k-IPC for $k \geq 3$. Since individual lifetime utilities are EDUs, k-IPC is equivalent to k-unanimity, and $U^k_{ij} = u_i + \delta_j u_i + \cdots + \delta_j^{k-1} u_i$. By Proposition A1 and previous arguments, there exist nonnegative λ_{ij} with $\sum_{ij} \lambda_{ij} = 1$ such that, for all $\mathbf{z} \in \mathcal{L}^k$,

$$u(z_1) + \delta u(z_2) + \cdots + \delta^{k-1} u(z_k) = \sum_{ij} \lambda_{ij} \Big(u_i(z_1) + \delta_j u_i(z_2) + \cdots + \delta_j^{k-1} u_i(z_k) \Big).$$

Now, let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, we obtain:

$$u = \sum_i \alpha_i u_i \quad \text{and} \quad \delta^k = \sum_j \gamma_j \delta_j^k \quad \text{for all } k.$$

Therefore, we must have $\gamma_j = 0$ or 1, i.e., the rule is dictatorial for the conception of the social discount factor.

D.7 Proof of Corollary 2

The necessity part is immediate.

Sufficiency part. Suppose that society satisfies k-IPC, for $k = 1$. Therefore, 1-IPC is equivalent to k-unanimity for $k = 1$ and $U^1_{ij} = u_i$. Therefore, Proposition A1 implies the existence of α_i with $\sum_i \alpha_i = 1$ such that $u = \sum_i \alpha_i u_i$.

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