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POROSITY IN THE SPACE OF HÖLDER-FUNCTIONS.

MOHAMMED BACHIR

ABSTRACT. Let (X, d) be a bounded metric space with a base point 0_X , $(Y, \|\cdot\|)$ be a Banach space and $\text{Lip}_0^\alpha(X, Y)$ be the space of all α -Hölder-functions that vanish at 0_X , equipped with its natural norm ($0 < \alpha \leq 1$). Let $0 < \alpha < \beta \leq 1$. We prove that $\text{Lip}_0^\beta(X, Y)$ is a σ -porous subset of $\text{Lip}_0^\alpha(X, Y)$, if (and only if) $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$ (i.e. d is non-uniformly discrete). A more general result will be given.

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1. INTRODUCTION

The main result of this note is Theorem 1, which gives a condition for some class of subsets of Lipschitz functions to be σ -porous subsets. The result in the abstract, as well as all the other results of this note, are just a very immediate consequence of this main result. However, the main motivations which led to the main theorem of this note, was precisely the result mentioned in the abstract.

Given a metric space (X, d) with a distinguished point 0_X (called a base point of X) and a Banach space $(Y, \|\cdot\|)$, we denote by $\text{Lip}_0(X_d, Y)$ (or by $\text{Lip}_0(X, Y)$, if no ambiguity arises) the Banach space of all Lipschitz functions from X into Y that vanish at the base point 0_X , equipped with its natural norm defined by

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(x')\|}{d(x, x')} : x, x' \in X; x \neq x'\right\}, \forall f \in \text{Lip}_0(X_d, Y).$$

We denote simply $\text{Lip}_0(X_d)$ or $\text{Lip}_0(X)$, if $Y = \mathbb{R}$. The space $L(X, Y)$ denotes the space of all linear bounded operators from X into Y . The space X^* denotes the topological dual of X . Notice that the space $\text{Lip}_0(X, Y)$ can be isometrically identified to $L(\mathcal{F}(X), Y)$ where $\mathcal{F}(X)$ is the free-Lipschitz space over X introduced by Godefroy-Kalton in [2]. Let us recall the definition of σ -porosity.

Definition 1. Let (F, d) be a metric space and A be a subset of F . A set A of F is called porous if there is a $c \in (0, 1)$ so that for every $x \in A$ there are $(y_n) \subset F$ with $y_n \rightarrow x$ and so that $B(y_n, cd(y_n, x)) \cap A = \emptyset$ for every n (We denote by $B(z, r)$ the closed ball with center z and radius r). A set A is called σ -porous if it can be represented as a union $A = \cup_{n=0}^{+\infty} A_n$ of countably many porous sets (the porosity constant c_n may vary with n).

Every σ -porous set is of first Baire category. Moreover, in \mathbb{R}^n , every σ -porous set is of Lebesgue measure zero. However, there does exist a non- σ -porous subset of \mathbb{R}^n which is of the first category and of Lebesgue measure zero (for more informations about σ -porosity, we refer to [7] and [5]).

The property (\mathcal{P}) . Let (X, d) be a metric space and Y be a Banach space. Let F be a nonempty (closed) convex cone of $\text{Lip}_0(X, Y)$. We say that F satisfies property (\mathcal{P}) if there exists a positive constant $K_F > 0$ depending only on F such that:

$$(\mathcal{P}) \quad \forall (x, x') \in X \times X, \exists p \in F : \|p\|_L \leq K_F \text{ and } \|p(x) - p(x')\| = d(x, x').$$

This property is related to the Hahn-Banach theorem and norming sets.

Examples 1. The property (\mathcal{P}) satisfied in the following cases:

(i) if X is a normed space and F contains the space $X^*.e := \{x \mapsto p(x).e : p \in X^*\}$, where $e \in Y$ is a fixed point such that $\|e\| = 1$.

(ii) if (X, d) is a metric space and F contains the functions $d_z : x \mapsto d(x, z).e$, for all $z \in X$, where $e \in Y$ is a fixed point is such that $\|e\| = 1$.

(iii) In particular, the space $\text{Lip}_0(X, Y)$ satisfies the property (\mathcal{P}) . If moreover, X is a normed space, then $L(X, Y)$ has the property (\mathcal{P}) too.

Proof. (i) By the Hahn-Banach theorem, for all $x \in X$ there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Then, for each $x \in X$, we consider the continuous linear map $p_x = x^*.e : X \rightarrow Y$ defined by $p_x(z) = x^*(z)e$ for all $z \in X$, and the property (\mathcal{P}) is satisfied.

(ii) Immediat.

(iii) This part follows from (i) and (ii) respectively. \square

2. THE MAIN RESULT

We are going to give the proof of the main result of this note. Let (X, d) be a metric space with a base point 0 and Y be a Banach space. Let $F \subset \text{Lip}_0(X, Y)$ and $\phi : X \times X \rightarrow \mathbb{R}^+$ be a positive function such that $\phi(x, x') = 0$ if and only if $x = x'$. For each real number $s > 0$, we denote:

$$\mathcal{N}_{\phi, s}(F) := \{f \in F : \sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \leq s\},$$

$$\mathcal{N}_{\phi}(F) := \{f \in F : \sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} < +\infty\}.$$

Notice that $\mathcal{N}_\phi(F) = \cup_{k \in \mathbb{N}} \mathcal{N}_{\phi, k}(F)$ and $\mathcal{N}_\psi(F) \subset \mathcal{N}_\phi(F)$ if $\psi \leq \phi$.

Theorem 1. *Let F be a nonempty (closed) convex cone of $\text{Lip}_0(X, Y)$ satisfying (\mathcal{P}) . Let $\phi : X \times X \rightarrow \mathbb{R}^+$ be any positive function such that $\phi(x, x') = 0$ if and only if $x = x'$. Suppose that $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$, then for every positive real number $s > 0$, we have that $\mathcal{N}_{\phi, s}(F)$ is a porous subset of $(F, \|\cdot\|_L)$. Consequently, the following assertions are equivalent.*

- (1) $\mathcal{N}_\phi(F) \neq F$.
- (2) $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$.
- (3) $\mathcal{N}_\phi(F)$ is a σ -porous subset of $(F, \|\cdot\|_L)$.

Proof. (1) \implies (2). Suppose that $\alpha := \inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} > 0$, then $\phi(x, x') \geq \alpha d(x, x')$ for all $x, x' \in X$. It follows that for every $f \in F$, we have that

$$\sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \leq \|f\|_L \sup_{x, x' \in X; x \neq x'} \frac{d(x, x')}{\phi(x, x')} \leq \frac{\|f\|_L}{\alpha} < +\infty.$$

Thus, $\mathcal{N}_\phi(F) = F$. Part (3) \implies (1) is trivial.

Let us prove that if $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$, then for every $s > 0$, we have that $\mathcal{N}_{\phi, s}(F)$ is a porous subset of $(F, \|\cdot\|_L)$, this gives in particular (2) \implies (3). Indeed, if $\inf\{\frac{\phi(x, x')}{d(x, x')} : x, x' \in X; x \neq x'\} = 0$, then there exists a pair of sequences $(a_n), (b_n) \subset X$ such that $0 < r_n := \frac{\phi(a_n, b_n)}{d(a_n, b_n)} \rightarrow 0$. By assumption, there exists $K_F > 0$ and a sequence $(p_n) \subset F$ such that $\|p_n\|_L \leq K_F$ and $\|p_n(a_n) - p_n(b_n)\| = d(a_n, b_n)$, for all $n \in \mathbb{N}$. Let $f \in \mathcal{N}_{\phi, s}(F)$, then we have that $\sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \leq s$. It follows that

$$\begin{aligned} \frac{\|(f + \sqrt{r_n} p_n)(a_n) - (f + \sqrt{r_n} p_n)(b_n)\|}{\phi(a_n, b_n)} &\geq \sqrt{r_n} \frac{\|p_n(a_n) - p_n(b_n)\|}{\phi(a_n, b_n)} - \frac{\|-(f(a_n) - f(b_n))\|}{\phi(a_n, b_n)} \\ &\geq \sqrt{r_n} \frac{d(a_n, b_n)}{\phi(a_n, b_n)} - \sup_{x, x' \in X; x \neq x'} \frac{\|f(x) - f(x')\|}{\phi(x, x')} \\ &\geq \frac{1}{\sqrt{r_n}} - s \end{aligned}$$

Since, $r_n \rightarrow 0$, when $n \rightarrow +\infty$, there exists a subsequence (r_{n_m}) such that

$$\frac{1}{\sqrt{r_{n_m}}} > 4s, \quad \forall m \in \mathbb{N}.$$

We set $f_m = f + \sqrt{r_{n_m}} p_{n_m} \in F$, for all $m \in \mathbb{N}$. We have that

$$\|f_m - f\|_L = \sqrt{r_{n_m}} \|p_{n_m}\|_L \leq K_F \sqrt{r_{n_m}} \rightarrow 0 \text{ when } m \rightarrow +\infty.$$

Let us prove that $B(f_m, \frac{1}{2K_F}\|f_m - f\|_L) \subset F \setminus \mathcal{N}_{\phi,s}(F)$ for all $m \in \mathbb{N}$. Indeed, let $g \in B(f_m, \frac{1}{2}\|f_m - f\|_L)$, then we have using the above informations that

$$\begin{aligned}
\frac{\|g(a_{n_m}) - g(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} &\geq \frac{\|f_m(a_{n_m}) - f_m(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} - \frac{\|(f_m - g)(a_{n_m}) - (f_m - g)(b_{n_m})\|}{\phi(a_{n_m}, b_{n_m})} \\
&\geq \left(\frac{1}{\sqrt{r_{n_m}}} - s\right) - \|f_m - g\|_L \frac{d(a_{n_m}, b_{n_m})}{\phi(a_{n_m}, b_{n_m})} \\
&\geq \left(\frac{1}{\sqrt{r_{n_m}}} - s\right) - \frac{1}{2K_F} \|f_m - f\|_L \frac{d(a_{n_m}, b_{n_m})}{\phi(a_{n_m}, b_{n_m})} \\
&\geq \left(\frac{1}{\sqrt{r_{n_m}}} - s\right) - \frac{1}{2} \sqrt{r_{n_m}} \frac{1}{r_{n_m}} \\
&= \frac{1}{2\sqrt{r_{n_m}}} - s \\
&> s.
\end{aligned}$$

Thus, we have that $g \in F \setminus \mathcal{N}_{\phi,s}(F)$ and so that $B(f_m, \frac{1}{2}\|f_m - f\|_L) \subset F \setminus \mathcal{N}_{\phi,s}(F)$ for all $m \in \mathbb{N}$. Thus, $\mathcal{N}_{\phi,s}(F)$ is porous in F (with $c = \frac{1}{2K_F}$). It follows that $\mathcal{N}_{\phi}(F) = \cup_{k \in \mathbb{N}} \mathcal{N}_{\phi,k}(F)$ is σ -porous in $(F, \|\cdot\|_L)$. \square

2.1. Immediate consequences. We deduce immediately the result mentioned in the abstract.

Corollary 1. *Let $X_1 := (X, d_1)$ and $X_2 := (X, d_2)$ be a set equipped with two metrics such that $d_1 \leq d_2$ and let $(Y, \|\cdot\|)$ be a Banach space. Then, $\text{Lip}_0(X_1, Y)$ is a σ -porous subset of $\text{Lip}_0(X_2, Y)$ if (and only if) d_1 and d_2 are not equivalent, if and only if $\text{Lip}_0(X_1, Y) \neq \text{Lip}_0(X_2, Y)$.*

Proof. We use Theorem 1 and part (iii) of Exemple 1 observing the following equality $\text{Lip}_0(X_1, Y) = \mathcal{N}_{d_1}(\text{Lip}_0(X_2, Y))$. \square

Notice that if $0 < \alpha \leq 1$ and d is a metric, so is d^α , hence the above corollary applies to the space of α -Hölder-functions that vanish at 0_X which is $\text{Lip}_0^\alpha(X, Y) := \text{Lip}_0(X_{d^\alpha}, Y)$. Notice also that if $0 < \alpha < \beta \leq 1$ and d is bounded, then $\text{Lip}_0^\beta(X, Y) \subset \text{Lip}_0^\alpha(X, Y)$. The metrics d^α and d^β are not equivalent if and only if, $\inf\{\frac{d^\beta(x, x')}{d^\alpha(x, x')} : x, x' \in X; x \neq x'\} = 0$, if and only if $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$ (since $\beta > \alpha$). Thus, we get the result of the abstract.

Corollary 2. *Let (X, d) be a bounded metric space with a base point 0_X , $(Y, \|\cdot\|)$ be a Banach space and $0 < \alpha < \beta \leq 1$. Then, $\text{Lip}_0^\beta(X, Y)$ is a σ -porous subset of $\text{Lip}_0^\alpha(X, Y)$, if and only if $\inf\{d(x, x') : x, x' \in X; x \neq x'\} = 0$.*

For the linear case, we have the following results.

Corollary 3. *Let $X_1 := (X, \|\cdot\|_1)$ and $X_2 := (X, \|\cdot\|_2)$ be a linear space equipped with two norms such that $\|\cdot\|_1 \leq \|\cdot\|_2$ and let $(Y, \|\cdot\|)$ be a Banach space. Then, $L(X_1, Y)$ is a σ -porous subset of $L(X_2, Y)$ if and only if $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent if and only if $L(X_1, Y) \neq L(X_2, Y)$.*

Proof. We use Theorem 1 and part (iii) of Exemple 1 after observing that $L(X_1, Y) = \mathcal{N}_{\|\cdot\|_1}(L(X_2, Y))$. \square

Example 1. Let $i : (l^1(\mathbb{N}), \|\cdot\|_1) \rightarrow (l^1(\mathbb{N}), \|\cdot\|_\infty)$ be the continuous identity map. Then the image of the adjoint i^* of i is a σ -porous subset of $(l^\infty(\mathbb{N}), \|\cdot\|_\infty)$.

We give in the following corollary a connexion between the surjectivity of the adjoint T^* of a one-to-one bounded linear operator T and the non- σ -porosity of its image (see in this sprit, the open mapping theorem in [6, Theorem 2.11]).

Proposition 1. *Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Let $T : X \rightarrow Z$ be a one-to-one bounded linear operator and T^* its adjoint. Then, the following assertions are equivalent.*

- (i) $T^*(Z^*)$ is not contained in a σ -porous subset of X^* .
- (ii) There exists $\alpha > 0$ such that $\alpha\|x\|_X \leq \|T(x)\|_Z$ for all $x \in X$.
- (iii) T^* is onto.

Proof. Since $T : X \rightarrow Z$ is a one-to-one bounded linear operator, then, the following map define another norm on X :

$$\|x\| := \frac{\|T(x)\|_Z}{\|T\|} \leq \|x\|_X, \quad \forall x \in X.$$

Let us denote $X_1 := (X, \|\cdot\|)$. By Corollary 3, applied with $Y = \mathbb{R}$, we have that X_1^* is a σ -porous subset of X^* if and only if $\|\cdot\|$ and $\|\cdot\|_X$ are not equivalent. Thus, if (ii) is not satisfied (that is, $\|\cdot\|$ and $\|\cdot\|_X$ are not equivalent) then, since $T^*(Z^*) \subset X_1^*$ we get that $T^*(Z^*)$ is contained in a σ -porous subset of X^* . Hence, (i) \implies (ii) is proved. Now, suppose that (ii) holds, it follows that $T(X)$ is closed in Z . Let $x^* \in X^*$ and define ϕ on $T(X)$ by $\phi(T(x)) := x^*(x)$ for all $x \in X$. Clearly ϕ is well defined (since T is one-to-one) and linear continuous on $T(X)$. Thus, ϕ extends to a linear continuous functional $y^* \in Z^*$ and we have $T^*(y^*) = y^* \circ T = x^*$. Hence, T^* is onto and (ii) \implies (iii) is proved. Part (iii) \implies (i), is trivial. \square

Let $(X, \|\cdot\|)$ be a normed space, and let S be a nonempty subset of the dual space X^* . The set S is called separating if: $x^*(x) = 0$ for all $x^* \in S$ implies that $x = 0$. It is called norming if the functional

$$N_S(x) = \sup_{x^* \in S; x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|},$$

is an equivalent norm on X (see [2] for the use of this notion).

Proposition 2. *Let $(X, \|\cdot\|)$ be a normed space. Every separating subset $S \subset X^*$ which is not contained in a σ -porous subset of X^* , is norming.*

Proof. It is clear that $N(x) \leq \|x\|$ for all $x \in X$. On the other hand, we have that

$$S \subset \mathcal{N}_{N_S}(X) := \{x^* \in X^* : \sup_{N_S(x)=1} |x^*(x)| < +\infty\}.$$

Since S is not contained in a σ -porous subset of X^* , then $\mathcal{N}_{N_S}(X)$ must be non- σ -porous, which implies from Theorem 1 that $\inf_{\|x\|=1} N_S(x) > 0$. Hence N_S is equivalent to $\|\cdot\|$. \square

2.2. Application to the barrier cone and polar of sets. Let X be a normed space and K be a nonempty subset of X . The barrier cone of K is the subset $B(K)$ of the topological dual X^* defined by

$$B(K) = \{x^* \in X^* : \sup_{x \in K} x^*(x) < +\infty\}.$$

The polar set of K is a subset of the barrier cone of K defined as follows:

$$K^\circ = \{x^* \in X^* : \sup_{x \in K} x^*(x) \leq 1\}.$$

The study of barrier cones has interested several authors. It is shown in [4, Theorem 3.1.1] that for a closed convex subset K of X , we have that $B(K) = X^*$ if and only if K is bounded, on the other hand, $B(K)$ is dense in X^* if and only if K does not contain any halfline. In general, we know that $\overline{B(K)} \neq X^*$ (see example in [1]). A study of the closure of the barrier of a closed convex set is given in [1]. As an immediate consequence of main theorem, we obtain below that the barrier cone of some general class of unbounded subsets is a σ -porous subset of the dual X^* . This shows that in general, the barrier cone may be a "very small" subset of X^* .

We define the class $\Phi(X)$ of positive functions on X (not necessarily continuous) as follows: $\phi \in \Phi(X)$ if and only if, $\phi : X \rightarrow \mathbb{R}^+$ and satisfies

- (i) $\phi(\lambda x) = |\lambda|\phi(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$
- (ii) $\phi(x) = 0$ if and only if $x = 0$

For every $\phi \in \Phi(X)$, we denote $S_\phi := \{x \in X : \phi(x) = 1\}$ and $C_\phi := \{x \in X : \phi(x) \leq 1\}$. Notice, that in general C_ϕ is not a convex set (resp. not closed), if we do not suppose that ϕ is a convex function (resp. a continuous function). It is easy to see that

$$(1) \quad C_\phi \text{ is bounded} \iff S_\phi \text{ is bounded} \iff \inf_{x \in X : \|x\|=1} \phi(x) > 0.$$

Thanks to the symmetry of $\phi \in \Phi(X)$ and the fact that $B(C_\phi) = B(S_\phi)$, we have using the notation of Theorem 1, that

$$(2) \quad B(C_\phi) = \mathcal{N}_\phi(X^*).$$

The polar of S_ϕ coincides with $\mathcal{N}_{\phi,1}(X^*)$ and we have

$$(3) \quad C_\phi^\circ \subset S_\phi^\circ = \mathcal{N}_{\phi,1}(X^*).$$

Now, using (1), (2) and (3) and applying Theorem 1 to the spaces $F = X^*$ and $Y = \mathbb{R}$ (using Exemple 1), we get directly the following informations about the size of the barrier cone, as well as the polar of a set of the form C_ϕ in the dual space.

Corollary 4. *Let X be a normed space and $\phi \in \Phi(X)$. If C_ϕ is not bounded in X , then the polar C_ϕ° is contained in a porous subset of X^* . Moreover, the followin assertions are equivalent.*

- (i) $B(C_\phi) \neq X^*$.
- (ii) C_ϕ is not bounded in X .
- (iii) $B(C_\phi)$ is a σ -porous subset of X^* .

We deduce that, if K is any nonempty subset of X such that $S_\phi \subset K$, for some $\phi \in \Phi(X)$ with S_ϕ not bounded, then the polar K° is contained in a porous subset of X^* and the barrier cone $B(K)$ is contained in a σ -porous subset of X^* . Notice that if K is a closed absorbing disk in X , does not contain a non-trivial vector subspace and is a neighborhood of the origin in X , then the Minkowski functional ϕ_K of K is a continuous norm (with respect to the norm $\|\cdot\|$, but not equivalent to it, if K is not bounded) hence $\phi_K \in \Phi(X)$ and we have that $K = C_{\phi_K}$.

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