# Economic Dynamics with Renewable Resources and Pollution*

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# Abstract

This article considers a two-sector economy with externalities. In particular, the analysis involves an industrial sector whose polluting production activities have negative effects on the regeneration of a natural resource in the other sector. Without convexity or supermodularity, we prove that the economy evolves to increase the *net gain of stock* (a similar notion to the *net gain of investment* in Kamihigashi & Roy [10]), and establish the conditions ensuring the convergence of the economy in the long run.

**Keywords.** Two-sector economy, renewable resources, pollution externality, Ramsey model.

# Contents

# 1 Introduction

Natural resources play an important role in the economy. Intriguingly, natural resources are not always a boon to economic growth. While abundant resources may help a country overcome the fixed cost problem and avoid the poverty trap (see Le Van & al [12]), they might induce an economy to consume beyond its means, potentially leading to stagnation in the long run (see Rodiguez & Sachs [14], Elisson & Turnovsky [8]).

The existing literature has explored the impact of natural resources in presence of externalities in a multisector economy. In particular, consider an economy with an industrial production sector and a natural resource exploitation sector (such as forestry or fishery). While the natural resource may enhance the productivity of the production sector or provide an additional source of income to the representative household, the production sector typically engages in polluting industrial activities at the detriment of the renewable resource, as has been studied by Beltratti & al [4], and Ayong Le Kama [3]. These authors consider the renewable resource as a consumption good as well as a production input. The regenerating capacity of the resource is impaired by pollution from the final good sector. Under suitable conditions, the existence of a stationary state and its local stability are proved.

This approach is appealing, but as Wirl [19] has observed, there is always room for limit cycles. Multiple long-run outcomes exist and are separated by a threshold, even under the convexity of the model. In this paper, we propose a new approach to study a two-sector economy with a renewable resource under discrete time configuration. We specify the conditions that ensure long-run convergence of the economy. Our approach can be applied not only to the work of Beltratti & al [4] and Ayong Le Kama [3], but also for other multisector models.

We consider a two-sector economy with an industrial sector that uses intermediate inputs to produce a final consumption good, and another sector, called the exploitation sector, which engages in exploiting a renewable resource. This resource can be sold directly at an exogenously determined market price. We assume there is an infinitely-lived representative consumer who allocates total incomes between consumption and capital investment to maximize intertemporal utility. She can use the income generated from the sales of the natural resource to invest in physical capital or to purchase consumption good.

This problem is challenging since we cannot follow the standard techniques laid out in the dynamic programming literature to study the long-term behavior of the economy. Usually, as well presented in Stokey & Lucas (with Prescott) [17], the Euler equations provide us with information on the optimal choice of investment and exploitation. In our economy, analyzing the Euler equations might not be appropriate since we are not sure whether the optimal choice belongs to the interior of the domain of definition. Moreover, since supermodularity is violated due to the indirect utility function having negative crossed derivatives, we cannot apply the techniques of Amir [1].[1]

To overcome this difficulty, we introduce the concept *net gain of stock*, which is the difference between the discounted value of production, and the existing resource stock and capital.[2] This concept is similar to the *net gain of investment* presented by Majumdar & Nermuth [13], Dechert & Nishimura [6], Mitra & Ray [16] or Kamihigashi & Roy [10]. As we shall see, the analysis of the *net gain of investment* can help illuminate our understanding of economic dynamics. Following Kamihigashi & Roy [10], we prove that the economy evolves to increase the value of the *net gain of stock* sometime in the future. This property has an important implication. It ensures that in the long run, the economy gets very close a steady state[3]. Furthermore, we specify conditions for the uniqueness of the steady states, and for the convergence of the economy in the long run.

The rest of the article is organised as follows. Section 2 considers the problem of the representative consumer without the negative externality of the production sector

[1]For the definition and a detailed survey on the supermodular economy, see Amir [1][2].

[2]For example in a one-dimensional economy, given the discount factor $\beta$, the production function $f$ and capital stock $k$, the *net gain of stock* is equal to $\beta f(k) - k$.

[3]If the steady state is unique, then convergence is ensured.

on the exploitation sector. Section 3 takes into account the negative impact of the polluting industrial sector on the regenerating capacity of the other sector. This chapter contains the main results of our paper, including the characterization of the conditions for the uniqueness of the steady states, and the long-run convergence of the economy. All proofs are given in the appendix.

# 2 Model without emission

## 2.1 Fundamentals

We consider a two-sector economy, one engaged in industrial activities to produce a final consumption good, and the other in the exploitation of a renewable resource.

The industrial sector is characterized by a production function $f$ which satisfies the standard properties such as monotonicity, concavity and the Inada conditions. To simplify the exposition, we assume without loss of generality (WLOG) that physical capital depreciates fully after each period.

The exploitation sector is characterized by the regenerating function $\eta$ and the price of the renewable resource $\theta > 0$, which is assumed exogenous[4]. We assume in this section that the function $\eta$ depends only on the natural resource stock and not on the industrial activities. In other words, pollution from industrial activities have no effect on the renewable resource.

At the beginning of period $t$, the economy has capital stock $k_t$ and renewable resource stock $y_t$, which generate an output from production $f(k_t)$ and a regenerated stock $\eta(y_t)$ of the resource, respectively. Let $x_t$ denote the amount of natural resource exploited by the consumer and $R_t$ the total revenue available to her at the beginning of period $t$. Clearly $R_t = f(k_t) + \theta x_t$. She then decides to allocate this revenue between current consumption $c_t$ and next-period investment in physical capital $k_{t+1}$. Given the initial capital and natural resource stocks $k_0$ and

[4]The case where $\theta$ varies in time or is a function of the resource stock is interesting, but the analysis is much more complex. This can be a subject for further research.

$y_0$, respectively, the representative consumer solves the intertemporal optimization program:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$c_t + k_{t+1} \leq f(k_t) + \theta x_t,$$

$$y_{t+1} = \eta(y_t) - x_t,$$

$$c_t, k_t, x_t, y_t \geq 0 \text{ for any } t \geq 0,$$

where $\beta \in (0,1)$ is the discount factor. Replacing $x_t$ with $\eta(y_t) - y_{t+1}$, we can rewrite the problem as:

$$v(k_0, y_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$c_t + k_{t+1} + \theta y_{t+1} \leq f(k_t) + \theta \eta(y_t),$$

$$y_{t+1} \leq \eta(y_t),$$

$$c_t, k_t, y_t \geq 0 \text{ for any } t.$$

Observe from the first constraint that with the natural resource as an additional source of revenue, the capital stock $k_{t+1}$ can be greater than the output $f(k_t)$ generated by the industrial sector. The second constraint says that the resource available tomorrow comes only from the regeneration of today's natural stock.

For each $(k, y) \in \mathbb{R}^2_+$, define

$$\Gamma(k, y) = \{(k', y') \in \mathbb{R}^2_+ \text{ such that } k' + \theta y' \leq f(k) + \theta \eta(y) \text{ and } y' \leq \eta(y)\}.$$

A sequence $\{(k_t, y_t)\}_{t=0}^{\infty}$ is feasible from $(k_0, y_0)$ if $\forall t \geq 0$, $(k_{t+1}, y_{t+1}) \in \Gamma(k_t, y_t)$. Let $\Pi(k_0, y_0)$ denote the set of all feasible paths $\{(k_t, y_t)\}_{t=0}^{\infty}$ from $(k_0, y_0)$.

We now impose standard conditions on the utility function, the production function and the resource regenerating function of the model.

**Assumption H1.** i) *The utility function* $u : \mathbb{R}_+ \to \mathbb{R}$ *is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada condition* $u'(0) = +\infty$.

ii) *The production function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *strictly increasing, strictly concave, continuously differentiable and satisfies* $f(0) = 0, f'(\infty) < 1, f'(0) = \infty$.

iii) *The regenerating function of the renewable resource* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing, strictly concave, continuously differentiable and satisfies* $\eta(0) = 0, \eta'(\infty) < 1, \eta'(0) = \infty$.

iv) *For any* $(k_0, y_0) \in \mathbb{R}^2_+$, *there exists a feasible sequence* $\{(k_t, y_t)\}_{t=0}^{\infty}$ *such that*

$$\sum_{t=0}^{\infty} \beta^t u\big(f(k_t) + \theta\eta(y_t) - k_{t+1} - \theta y_{t+1}\big) > -\infty.$$

These conditions are standard in the literature. They ensure that in the product topology, the set of feasible paths $\Pi(k_0, y_0)$ is compact and the value function $v$ is upper semi-continuous. It is well-established that under theses properties, an optimal path exists. In absence of externality, the concavity of the production function and the regenerating function ensures the uniqueness of the optimal path. Moreover, we can write the Bellman functional equation which admits $v$ as a solution[5].

The correspondence $\Gamma$ is non-empty, convex, compact-valued, and continuous. The value function, which is strictly increasing and strictly concave, is a solution to the Bellman functional equation[6]. The optimal policy function is well-defined and satisfies usual continuity properties. Readers interested in the proof of Proposition 2.1 can refer to the construction in the classical work of Stockey & Lucas (with Prescott) [17], Chapter 4[7].

PROPOSITION 2.1. *Assume* ***H1***.

i) *The correspondence* $\Gamma$ *is non-empty, convex, compact-valued, and continuous on* $\mathbb{R}^2_+$

[5]For the details, see Dana & Le Van [5] or Le Van & Morhaim [11].

[6]When the utility function is bounded from below, it is unique.

[7]In orther to apply the results in Stockey & Lucas (with Prescott) [17], one can use the following indirect utility function: $V(k, y, k', y') = u\big(f(k) + \theta\eta(y) - k' - \theta y'\big)$.

ii) *The value function* $v$ *satisfies the Bellman functional equation:*

$$v(k,y) = \max_{(k',y')\in\Gamma(k,y)} \left[u\big(f(k) + \theta\eta(y) - k' - \theta y'\big) + \beta v(k',y')\right].$$

iii) *There exists an upper semi-continuous policy function* $\varphi$ *such that*

$$\varphi(k,y) = \operatorname*{argmax}_{(k',y')\in\Gamma(k,y)} \left[u\big(f(k) + \theta\eta(y) - k' - \theta y'\big) + \beta v(k',y')\right].$$

iv) *The feasible sequence* $\{(k_t, y_t)\}_{t=0}^{\infty}$ *is optimal if and only if for any* $t$,

$$(k_{t+1}, y_{t+1}) = \varphi(k_t, y_t).$$

v) *Assume* $k_0 > 0$ *and* $y_0 > 0$. *The optimal path* $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ *satisfies the property that* $k_t^* > 0$, $y_t^* > 0$ *for any* $t \geq 0$.

Denote by $(k^s, y^s)$ the stocks such that

$$f'(k^s) = \frac{1}{\beta} \text{ and } \eta'(y^s) = \frac{1}{\beta}.$$

It is easy to verify that $(k^s, y^s)$ belongs to $\Gamma(k^s, y^s)$ and is the unique steady state.

## 2.2 Local and global dynamics

The dynamics of the economy under consideration is difficult to study since even though the Inada conditions are satisfied, we can not exclude the possibility that $y_{t+1}^* = \eta(y_t^*)$ for some date $t$. This prevents us from analysing the Euler equations directly or following well-known approaches in dynamic programming theory. Moreover, the violation of supermodularity (due to the indirect utility function having negative crossed derivatives) renders inapplicable the monotonicity results in Amir [1].

We tackle this issue by reformulating the problem as follows. For each $z > 0$, let

$$F(z) = \max_{k+\theta y = z} \left(f(k) + \theta\eta(y)\right). \tag{2.1}$$

We have the following lemma.

Lemma **2.1.** *The function $F$ defined in* (2.1) *is strictly concave. Moreover, with*

$$(k^z, y^z) = \underset{k+\theta y = z}{\operatorname{argmax}} \left( f(k) + \theta\eta(y) \right),$$

*we have* $0 < k^z < z$ *and* $0 < y^z < \dfrac{z}{\theta}$. *The derivatives satisfy* $f'(k^z) = \eta'(y^z) = F'(z)$.

The proof of Lemma 2.1 is immediate from Theorem 5.4, page 33 in Rockafellar [15]. Define

$$S = f(k_0) + \theta\eta(y_0). \tag{2.2}$$

Since $F$ is continuous and strictly increasing, it is invertible. Let $z_0 = F^{-1}(S)$, then $z_0$ is well-defined[8]. Consider the modified problem:

$$\max \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right],$$

$$\text{s.t. } c_t + z_{t+1} \leq F(z_t) \text{ for } t \geq 0,$$

$$z_0 \text{ given }.$$

Observe that thanks to Lemma 2.1, the modified problem is convex. It has a unique optimal path, which converges monotonically to the steady state $z^s$, the solution to $F'(z) = \frac{1}{\beta}$. We can verify that $z^s = k^s + \theta y^s$. For the optimal solution $\{z_t^*\}_{t=0}^{\infty}$ of the modified problem, define the corresponding path $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ by

$$(\tilde{k}_t, \tilde{y}_t) = \underset{k+\theta y = z_t^*}{\operatorname{argmax}} \big( f(k) + \theta\eta(y) \big).$$

Note that in general the corresponding path $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ may not satisfy the constraint $\tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$. The following lemma provides the conditions for the equivalence between the initial and the modified problems.

Lemma **2.2.** *Assume **H1**. The modified problem has a unique solution. Moreover,*

i) *Consider the solution $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ to the initial problem. Define*

$$z_0 = F^{-1}(f(k_0) + \theta\eta(y_0))$$

$$z_t^* = k_t^* + \theta y_t^*.$$

[8]This consideration is necessary, since $(k_0, y_0)$ may differ from $\operatorname{argmax}_{k+\theta y = z_0} (f(k) + \theta\eta(y))$.

*If* $0 < y^*_{t+1} < \eta(y^*_t)$ *for any* $t \geq 0$, *the sequence* $\{z^*_t\}_{t=0}^{\infty}$ *solves the modified problem.*

ii) *Consider the solution* $\{\tilde{z}_t\}_{t=0}^{\infty}$ *to the modified problem. For any* $t \geq 1$, *define*

$$(\tilde{k}_t, \tilde{y}_t) = \underset{k+\theta y=\tilde{z}_t}{\operatorname{argmax}} \left(f(k) + \theta\eta(y)\right).$$

*If* $0 < \tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$ *for any* $t \geq 0$, *the sequence* $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ *solves the initial problem.*

In other words, if the initial problem has an interior solution, then this solution is also the solution to the modified problem. If the modified problem generates a sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ such that $\tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$ for any $t$, then this sequence also solves the initial problem.

### 2.2.1 Local dynamics

The analysis of the modified problem allows us to study the local dynamics of the initial problem. If the economy begins near the steady state $(k^s, y^s)$, the constraint $\tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$ is satisfied and the sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ also solves the initial problem. Following Lucas & Stokey (and Prescott) [17], this economy converges geometrically to the steady state.

Proposition 2.2. *Assume* ***H1***. *Denote by* $z^s$ *the steady state of the modified problem and* $(k^s, y^s)$ *the steady state of the initial problem. We have:*

i) *The point* $(k^s, y^s)$ *satisfies*

$$(k^s, y^s) = \underset{k+\theta y=z^s}{\operatorname{argmax}} \left(f(k) + \theta\eta(y)\right).$$

ii) *There exists a neighborhood* $\mathcal{V}$ *of* $(k^s, y^s)$ *such that for any* $(k_0, y_0) \in \mathcal{V}$, *the optimal sequence* $\{(k^*_t, y^*_t)\}_{t=0}^{\infty}$ *for the initial problem converges to* $(k^s, y^s)$.

These local dynamic properties echo the results of the continuous-time analogs in the literature. They also prove extremely useful in the study of global dynamics, which we next examine.

### 2.2.2 Global dynamics

Let us now consider the case where the initial state $(k_0, y_0)$ is arbitrary. Since we may have $y^*_{t+1} = \eta(y^*_t)$ for some $t$, we cannot invoke Lemma 2.2. In other words, we cannot be sure that $(k^*_t, y^*_t)$ maximizes $f(k)+\theta\eta(y)$ under the constraint $k + \theta y = z^*_t$. Nevertheless, we can show that for $T$ sufficiently big, we have $0 < y^*_{t+1} < \eta(y^*_t)$ for any $t \geq T$. To do so, we introduce the important notion called the *net gain of stock* mentioned earlier in the paper. For each $(k, y) \in \mathbb{R}^2_+$, define

$$\Psi(k, y) = \beta\big(f(k) + \theta\eta(y)\big) - (k + \theta y). \tag{2.3}$$

This notion was first used in one-dimensional economics by Majumdar & Nermuth [13], Dechert & Nishimura [6] and Mitra & Ray [16] to study the properties of the steady states. Kamihigashi & Roy [9, 10] prove that the economy always evolves to increase the value of the net gain function in the future, otherwise we are at the steady state. Following their insight, we will prove that although the sequence of the *net gain of stock* may not rise monotonically, it will increase at some point in the future. This important property allows us to establish the long-run convergence of the economy.

The reasoning is as follows. Observe that $(k^s, y^s)$ is the maximizer of $\Psi(k, y)$. Suppose that the economy begins at an unsteady state, our goal is to prove that the *net gain of stock* must always increase in the future. In particular, we first prove that for any $t \geq 0$, there exists some date $t' > t$ such that $\Psi(k^*_{t'}, y^*_{t'}) > \Psi(k^*_t, y^*_t)$. Moreover, since

$$\sup_{t\geq 0} \Psi(k^*_t, y^*_t) = \Psi(k^s, y^s) = \sup_{(k,y)\in\mathbb{R}^2_+} \Psi(k, y),$$

there exists some period $t$ such that the state $(k^*_t, y^*_t)$ gets *very close* to the steady state. Then by virtue of Proposition 2.2, the optimal sequence converges rapidly to the steady state $(k^s, y^s)$ from this period.

These arguments are presented formally in Lemma 2.3, Lemma 2.4 and Proposition 2.3 below.

Lemma **2.3.** *Assume **H1**. The steady state is the only solution which maximizes* $\Psi$*:*

$$\operatorname*{argmax}_{(k,y)\in\mathbb{R}^2_+} \Psi(k,y) = \{(k^s, y^s)\}.$$

This lemma can be proved using the concavity of the functions $f$ and $\eta$. Lemma 2.4 is the most important intermediate result in the establishment of the long-term behaviour of the optimal path. It states that even though the sequence $\{\Psi(k_t^*, y_t^*)\}_{t=0}^{\infty}$ can be non-monotone, there exists some period in the future when the *net gain of stock* shall increase.

Lemma **2.4.** *Assume **H1**. Consider the initial state* $(k_0, y_0)$ *such that* $y_0 \leq \eta(y_0)$*. Exactly one of the following statements is true:*

i) *For any* $t$*,* $k_t^* = k_0$ *and* $y_t^* = y_0$*.*

ii) *There exists some* $t > 0$ *such that*

$$\Psi(k_t^*, y_t^*) > \Psi(k_0, y_0).$$

Lemma 2.4 tells us that for any unsteady initial state, the value of the *net gain of stock* will increase some day in future. The following proposition asserts that this value converges to $\Psi(k^s, y^s)$ and that the optimal path converges to $(k^s, y^s)$.

Proposition 2.3. *Assume **H1**. For any* $(k_0, y_0) \in \mathbb{R}^2_+$*, the optimal path converges to* $(k^s, y^s)$*.*

Let us now numerically illustrate the existence of a unique steady state and global convergence to this steady state. For simplicity, suppose that the utility function satisfies constant intertemporal elasticity of substitution (CIES)[9], and that both the production function and the resource generating function are Cobb-Douglas. In particular,

$$f(k) = Ak^{\alpha_k}, \quad \eta(k) = By^{\alpha_y}, \tag{2.4}$$

[9]This function has constant elasticity of marginal utility, that is $\frac{-u''(c)c}{u'(c)} = constant$. For details, see for example De la Croix & Michel [7], page 5.

where $A, B > 0$, $0 < \alpha_k, \alpha_y < 1$ along with other the model parameters are given in Table 1. The choice of $\alpha = 0.67$ reflects the common fact that the capital share in aggregate production functions is $\frac{2}{3}$.

| **Parameter** | **Value** |
|---|---|
| $\theta$ | 1 |
| $\beta$ | 0.98 |
| $\alpha_k$ | 0.67 |
| $\alpha_y$ | 0.8 |
| $A$ (TFP in the final good sector) | 2 |
| $B$ (TFP in the exploitation sector) | 1 |
| $k_0$ (Initial stock of physical capital) | $2k^s$ |
| $y_0$ (Initial stock of the renewable resource) | $0.2y^s$ |

Table 1: Parameters used for the simulated optimal paths when there is no emission

Notice that while varying the intertemporal elasticity of substitution (IES) alters neither the steady-state values nor the global convergence property, it affects the speed of convergence. In particular, the smaller the IES, the slower the optimal sequences converge to their corresponding steady states, as shown in Fig.1 and Fig.2.

# 3 Renewable resources and industrial externalities

## 3.1 Fundamentals

The analysis in Section 2 ignores the effect of the production sector on the regenerating capacity of the natural resource. In reality, the industrial activities can be polluting, creating negative externalities. In this paper, we hypothesise that

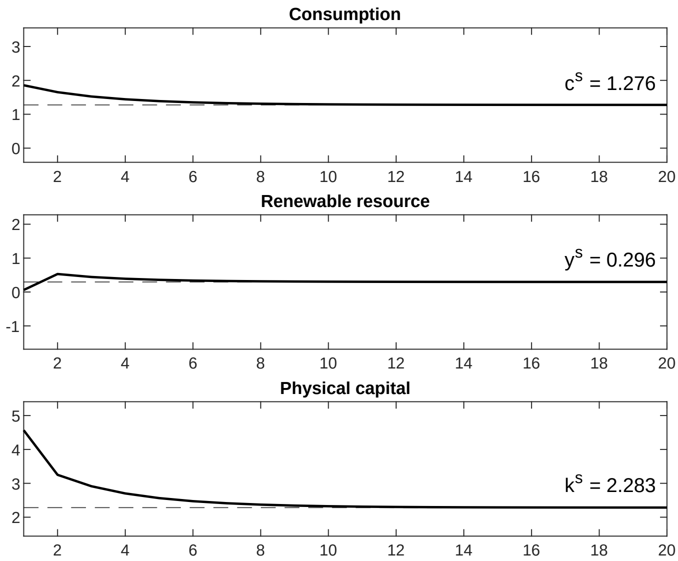


Figure 1: Optimal paths under no emission with constant intertemporal elasticity of substitution equal to 1 (logarithmic utility)

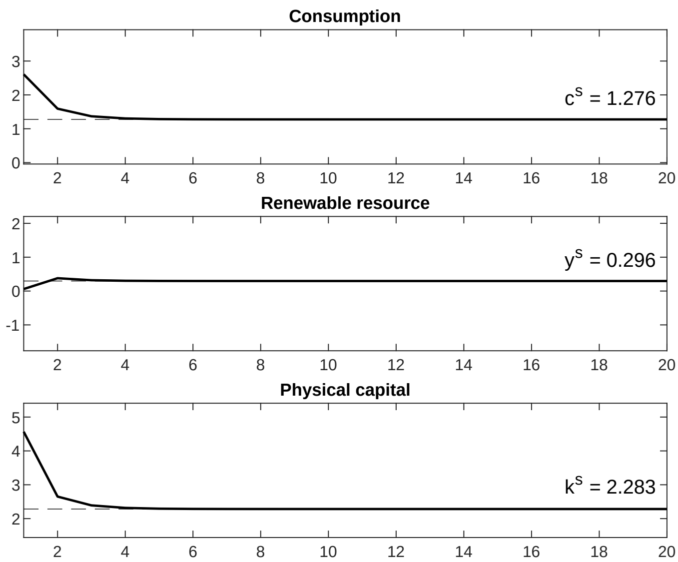


Figure 2: Optimal paths under no emission with constant intertemporal elasticity of substitution equal to 0.1

pollution from the production sector damages the replenishing capacity of the natural resource. The larger the scale of production, the more severe the negative externality.

To be concrete, let $E = E(k)$ denote the function of pollution caused by industrial activities. Since pollution is increasing in the scale of production, $E$ is strictly increasing in $k$. To capture the negative impact of pollution on resource regeneration, we let $\eta$, the regenerating function introduced in the previous section, depend negatively on $E$. Hence in this section $\eta$ takes two arguments $y$ and $E$: the growth rate of natural resource depends not only on the remain of its stock, but also on the environment in which it grows.

Observe that $\eta$ is not concave with respect to the second argument. Indeed, suppose the contrary, then $\eta(y, \cdot)$ is strictly decreasing for all $y$, implying that for $k$ sufficiently large, we obtain a negative value for the renewable resource, which is not intuitive.

**Assumption H2.** *Assume conditions* $(i)$ *and* $(ii)$ *in* ***H1****. Moreover, the following conditions are satisfied:*

i) *The function* $\eta : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is continuous and differentiable with respect to each argument. It is strictly increasing in the first and strictly decreasing in the second argument.*

ii) $\eta(0, E) = 0$ *for all* $E \geq 0$.

iii) *For any* $E > 0$, $\eta'(0, E) = \infty$ *and* $\eta'(\infty, E) < 1$.

iv) *For any* $(k_0, y_0) \in \mathbb{R}_+^2$, *there exists a feasible sequence* $\{(k_t, y_t)\}_{t=0}^{\infty}$ *such that*

$$\sum_{t=0}^{\infty} \beta^t u\big(f(k_t) + \theta\eta(y_t, E(k_t)) - k_{t+1} - \theta y_{t+1}\big) > -\infty.$$

The representative agent solves the intertemporal optimization program:

$$
\begin{aligned}
v(k_0, y_0) = & \max \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right], \\
& c_t + k_{t+1} + \theta y_{t+1} \leq f(k_t) + \theta \eta(y_t, E(k_t)), \\
& y_{t+1} \leq \eta \left( y_t, E(k_t) \right), \\
& c_t, k_t, y_t \geq 0 \text{ for any } t.
\end{aligned}
$$

A few remarks are in order. Observe that the model satisfies neither convexity nor supermodularity. Indeed, the fact that $\eta$ is not concave with respect to $E$ rules out convexity. The indirect utility function, defined on the domain of $(k_t, y_t, k_{t+1}, y_{t+1})$ is not supermodular since it does not satisfy the property that every cross derivative is positive, which is required in Amir [2]. The feasible correspondence remains compact-valued, and hence a solution exists. Yet it may not be unique; there might exist multiple optimal paths starting from the same initial state.

As in the previous section, let us first derive some basic properties of the value function and the optimal policy correspondence. For each $(k, y) \in \mathbb{R}^2_+$, define the feasible correspondence $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by:

$$
\Gamma(k, y) = \{(k', y') \in \mathbb{R}^2_+ \text{ such that } k' + \theta y' \leq f(k) + \theta \eta \left( y, E(k_t) \right) \text{ and } y' \leq \eta \left( y, E(k_t) \right) \}. \tag{3.1}
$$

The following proposition follows immediately from Stokey & Lucas (with Prescott) [17].

Proposition 3.1. *Assume* ***H2****.*

i) *The correspondence* $\Gamma$ *defined in* (3.1) *is continuous, convex, and compact-valued.*

ii) *The value function* $v$ *satisfies the Bellman functional equation:*

$$
v(k, y) = \max_{(k', y') \in \Gamma(k, y)} \left[ u\big( f(k) + \theta \eta \left( y, E(k) \right) - k' - \theta y' \big) + \beta v(k', y') \right].
$$

iii) *There exists an upper semi-continuous policy correspondence* $\varphi$ *defined by:*

$$\varphi(k,y) = \underset{(k',y')\in\Gamma(k,y)}{\operatorname{argmax}} \left[u\big(f(k) + \theta\eta\,(y, E(k)) - k' - \theta y'\big) + \beta v(k',y')\right].$$

iv) *A feasible sequence* $\{(k_t, y_t)\}_{t=0}^{\infty}$ *is optimal if and only if for any* $t$,

$$(k_{t+1}, y_{t+1}) \in \varphi(k_t, y_t).$$

v) *Assume that* $k_0 > 0$ *and* $y_0 > 0$. *If* $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ *is an optimal sequence, then* $k_t^* > 0$, *and* $y_t^* > 0$ *for any* $t \geq 0$.

## 3.2 Long-term dynamical analysis

### 3.2.1 Existence of the steady states

When the problem is not convex, the uniqueness of the steady states is not ensured. Let us describe some properties of the long-term behaviour of the economy. As in the previous section, define the *net gain of investment* function by:

$$\Psi^e(k,y) = \beta\,(f(k) + \theta\eta\,(y, E(k))) - (k + \theta y). \tag{3.2}$$

Define:

$$S^m = \underset{(k,y)\in\mathbb{R}_+^2}{\operatorname{argmax}} \left[\beta\,(f(k) + \theta\eta\,(y, E(k))) - (k + \theta y)\right]. \tag{3.3}$$

By the continuity of $f$ et $\eta$, it is easy to verify that $S^m \neq \emptyset$. Furthermore, for any $(k,y) \in S^m$, the constant sequence $\{k_t, y_t\}_{t=0}^{\infty}$ satisfying $(k_t, y_t) = (k,y)$ for all $t$, is feasible. The next proposition shows that any such $(k,y)$ is a steady state. Starting from any initial state which is not a steady state, the value of the *net gain of stock* will increase in the future.

Proposition 3.2. *Assume* ***H2***. *Then:*

i) *A steady state exists.*

ii) *Consider an initial state such that* $y_0 \leq \eta(y_0, E(k_0))$*. Either* $(k_0, y_0)$ *is a steady state, or for any optimal path* $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ *beginning from* $(k_0, y_0)$*, there exists some* $t \geq 0$ *such that*

$$\Psi^e(k_t^*, y_t^*) > \Psi^e(k_0, y_0).$$

As in Section 2, Proposition 3.2 allows us to prove that any optimal sequence must get very close a steady state at some point in the future. If there is only one steady state, then this state must be an absorbing point. That is, starting from anywhere in a neighborhood of this point, there exists an optimal path converging to it. By similar arguments to Section 2, we deduce that beginning from any initial state, there exists an optimal path converging to the steady state.

Note that although the possibility of multiple optimal paths can not be excluded, the set initial states which generate multiple optimal paths has zero measure (see Decher & Nishimura [6]). We can thus conclude that the economy converges *almost surely* in the long run.

#### 3.2.2 Uniqueness of the steady state and local dynamics

Let $\eta_1$ and $\eta_2$ be the partial derivatives of $\eta$ with respect to its first and the second argument, respectively.

**Assumption H3.** *The following system has unique solution:*

$$\begin{aligned} f'(k) + \theta\eta_2(y, E(k)) E'(k) &= \frac{1}{\beta}, \\ \eta_1(y, E(k)) &= \frac{1}{\beta}. \end{aligned}$$

Since this system of equations provides the necessary conditions for a steady state, Assumption **H3** ensures its uniqueness.

As in Section 2, we first analyse the dynamic that begins near the steady state. Define

$$G(z) = \max_{k+\theta y = z} \left[ f(k) + \theta\eta(y, E(k)) \right]. \tag{3.4}$$

Observe that $G$ is strictly increasing and differentiable. By Assumption **H3**, there exists a unique solution $z^s$ to $G'(z) = \frac{1}{\beta}$. By the Inada conditions, $G'(0) = \infty$ and $G'(\infty) < 1$. This implies that $G'(z) > \frac{1}{\beta}$ for $0 < z < z^s$ and $G'(z) < \frac{1}{\beta}$ for $z > z^s$[10].

Consider the following modified problem:

$$\max \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right],$$
$$\text{s.t. } c_t + z_{t+1} \leq G(z_t) \text{ for any } t \geq 0,$$
$$z_0 \text{ given }.$$

Since $G$ is strictly increasing, the indirect utility function $V(z, z') = u\left(G(z) - z'\right)$ has increasing differences. Following Amir [1], this implies the monotonicity of the optimal paths of the modified problem.

LEMMA **3.1.** *Assume **H2** and **H3**. Starting from any initial state $z_0$, every optimal path of the modified problem converges monotonically to the unique steady state $z^s$.*

As in Section 2, Lemma 3.1 allows us to describe the behaviour of the optimal path once the initial state is sufficiently close the steady state $(k^s, y^s)$. In the following proposition, we prove that starting from any initial state, there exists an optimal path converging to the steady state. The idea is that any optimal path must get "close" to the steady state, and from that new position, there is a path that converges monotonically to $(k^s, y^s)$.

PROPOSITION 3.3. *Assume **H2, H3**.*

i) *There exists a neighbourhood $\mathcal{V}$ of $(k^s, y^s)$ such that for any $(k_0, y_0) \in \mathcal{V}$, there is an optimal path starting from $(k_0, y_0)$ converging to $(k^s, y^s)$.*

[10] Since $G$ is differentiable, its derivatives satisfy the famous *Intermediate Value Property* (also known as the Darboux property, or Bolzano-Cauchy property), which states that if $G'(z) > \frac{1}{\beta}$ and $G'(z') < \frac{1}{\beta}$, then there exists some $\tilde{z}$ between $z$ and $z'$ such that $G'(\tilde{z}) = \frac{1}{\beta}$. Hence we do not need the continuity of $G'$.

ii) *For any* $(k_0, y_0)$*, there exists an optimal path starting from* $(k_0, y_0)$ *converging to* $(k^s, y^s)$*.*

#### 3.2.3 Global dynamics and long-term convergence

Let us first make the simplifying assumption that the pollution function is linear, so that $E(k) = \alpha k$, where $\alpha > 0$ captures the influence of the production sector on pollution (and consequently on the regeneration of the natural resource). Next, let us consider the plausible conditions to impose on $\eta$. We have argued above that $\eta$ is not concave with respect to the second argument. We can go a bit further to hypothesize that $\lim_{k\to\infty} \eta(y, \alpha k) = 0$, which is a reasonable assumption saying that when the scale of industrial production explodes, the overwhelming negative effect of pollution will wipe out the natural resource. This essentially means that $\eta(y, \cdot)$ is convex with respect to the second argument.

For simplicity let us assume that $\eta$ is separable:

$$\eta(y, E) = g(y)h(E). \tag{3.5}$$

**Assumption H4.** i) *The function* $g$ *is strictly increasing, strictly concave, and satisfies* $g'(0) = \infty$ *and* $g'(\infty) < 1$*.*

ii) *The function* $h$ *is strictly decreasing and convex.*

Observe that the "production function" of the modified problem, the function $G$ defined in (3.4), is strictly increasing but not necessarily concave. Define $k^m = \max\{\overline{k}, k_0\}$ where $\overline{k}$ is the solution to $f(k) = k$, and $y^m = \max\{\overline{y}, y_0\}$ where $\overline{y}$ is the solution to $g(y) = y$. Let $z^m = k^m + \theta y^m$. To ensure the concavity of $G$, we add the following mild assumption.

**Assumption H5.** *For any* $0 \le k \le z \le z^m$*, it holds:*

i)

$$f''(k) + \frac{1}{\theta} g''\left(\frac{z-k}{\theta}\right) h(\alpha k) - 2\alpha g'\left(\frac{z-k}{\theta}\right) h'(\alpha k) + \alpha^2 \theta g\left(\frac{z-k}{\theta}\right) h''(\alpha k) < 0.$$

ii)

$$\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k)-\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)<0.$$

Under Assumption **H5**, the function $G$ is strictly concave in $[0, z^m]$. The solution to $G'(z)=\frac{1}{\beta}$ is thus unique, verifying Assumption **H3**.

PROPOSITION 3.4. *Assume **H2**, **H4**, and **H5**. The steady state $(k^s, y^s)$ is unique and for any $(k_0, y_0)$, there exists an optimal path beginning from $(k_0, y_0)$ converging to $(k^s, y^s)$.*

Observe that by the concavity of $g$ and the convexity of $h$, Assumption **H5** is satisfied for sufficiently small $\alpha$ and $\theta$, implying that the economy converges in the long term in view of Proposition 3.4.

If the inequality in part $(i)$ of Assumption **H5** is satisfied without the presence of $f''(k)$, then $(i)$ implies $(ii)$ and we obtain the following corollary.

COROLLARY 3.1. *Assume **H2**, **H4**. Furthermore, assume that for any $0 \le k \le z \le z^m$, we have*

$$\frac{1}{\theta}g''\left(\frac{z-k}{\theta}\right)h(\alpha k)-2\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k)+\alpha^2\theta g\left(\frac{z-k}{\theta}\right)h''(\alpha k)<0.$$

*Then the steady state $(k^s, y^s)$ is unique and for any $(k_0, y_0)$, there exists an optimal path beginning from $(k_0, y_0)$ converging to $(k^s, y^s)$.*

Furthermore if $h$ is exponential satisfying $h(\alpha k)=e^{-\gamma\alpha k}$, then **H5** can be reduced to a simple condition on $g$.

**Assumption H6.** *For any $0 \le y \le y^m$, we have*

$$\frac{1}{\theta}g''(y)+2\alpha\gamma g'(y)+\alpha^2\gamma^2\theta g(y)<0.$$

Under **H6**, it is easy to verify that the conditions in **H5** are satisfied. Proposition 3.5 below is obtained as a direct consequence of Proposition 3.2.

PROPOSITION 3.5. *Consider the case $\eta(y,\alpha k)=g(y)e^{-\gamma\alpha k}$. Assume **H2**, **H4**, and **H6**. The steady state $(k^s, y^s)$ is unique and for any $(k_0, y_0)$, there exists an optimal path beginning from $(k_0, y_0)$ which converges to $(k^s, y^s)$.*

Let us illustrate the results with a numerical exercise. For simplicity assume that the utility function is logarithmic. The production function and the regeneration used for the simulation are Cobb-Douglass as in (2.4), and parameter values are given in Table 2.

| **Parameter** | **Value** |
|---|---|
| $\gamma$ | 0.5 |
| $\beta$ | 0.98 |
| $\alpha_k$ | 0.67 |
| $\alpha_y$ | 0.8 |
| $\alpha$ (Emission coefficient) | 0.2 |
| $A$ (TFP in final good sector) | 2 |
| $B$ (TFP in exploitation sector) | 1 |

Table 2: Parameters used for the numerical simulation in the case of exponential emission

The optimal paths of consumption, renewable resource and physical capital for two different values of resource's price are presented in Fig.3 and Fig.4. In both cases we start with an initial physical capital stock greater than the steady state and an initial resource stock lower than the steady state by the same fraction for convenient comparability. Observe that the higher the price of the resource, the greater the steady state values of consumption and the resource stock, and the smaller the steady state value of physical capital. The convergence speed also appears to be slower when the resource's price is higher.

Suppose now that the emission function takes the form $h(E) = (1+E)^{-\zeta}$. Assume logarithmic utility, Cobb-Douglas production and other parameters as in Table 2, we simulated the optimal paths for $\zeta = 0.5$ and $\zeta = 10$ in Fig.5 and Fig.6, respectively. Observe that $\zeta$ captures the impact of pollution on the renewable resource (while $\alpha$ reflects the intensity of industrial pollution). Notably, when the price of the natural resource is high and pollution has a large impact, physical capital is almost depleted at the steady state (Fig 6).

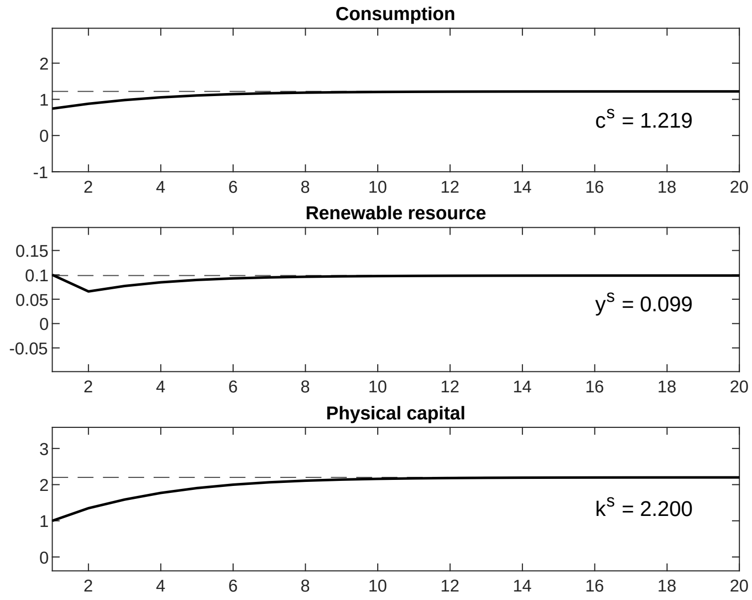


Figure 3: Optimal paths under exponential emission with low price of renewable resource $\theta = 1$

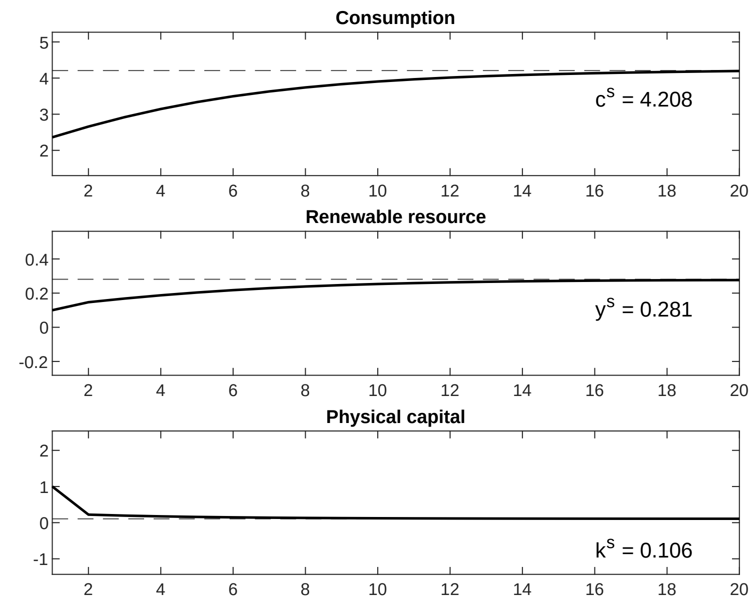


Figure 4: Optimal paths under exponential emission with high price of renewable resource $\theta = 50$

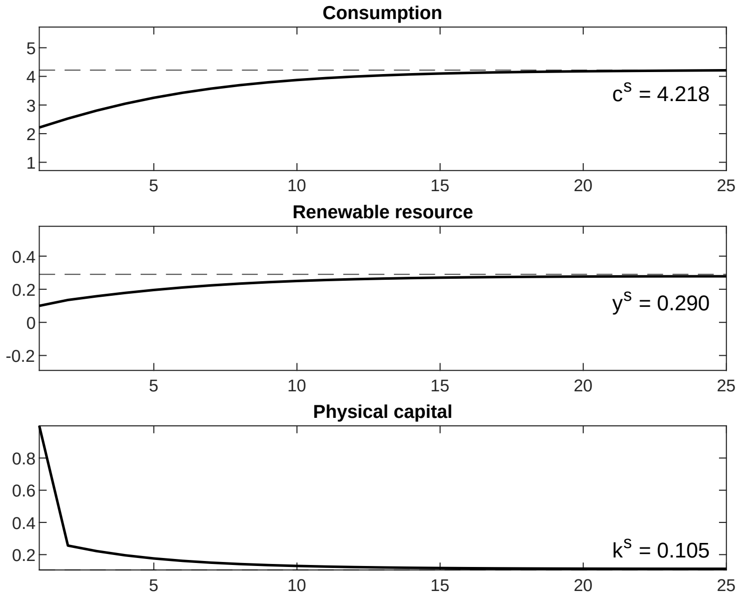


Figure 5: Optimal paths under non-exponential emission with $\zeta = 0.5$

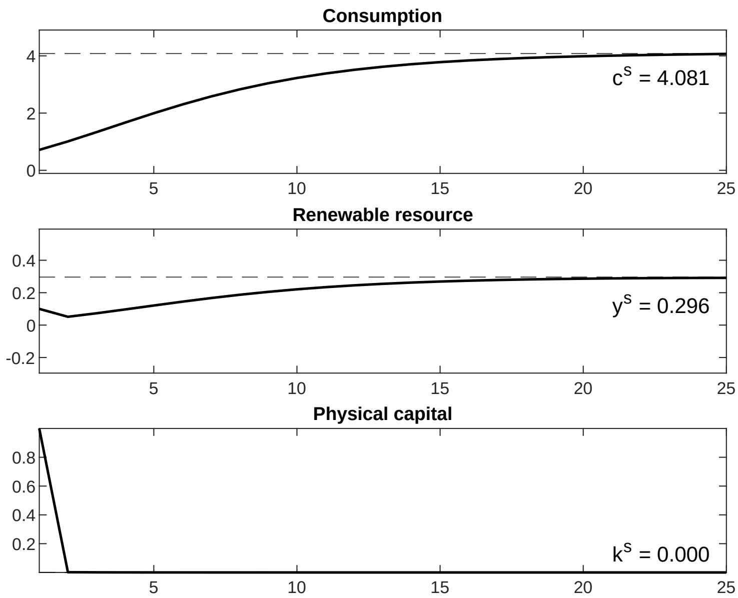


Figure 6: Optimal paths under non-exponential emission with $\zeta = 10$

# 4 Conclusion

In this article, in a configuration where the usual properties such as convexity or supermodularity are not satisfied, we develop a new method to analyse the long-term dynamics of the economy and prove that under suitable conditions, the economy converges in the long run. The simulation results suggest that the economy may exhibit some initial fluctuations, but then converges rapidly to the steady state.

# 5 Appendix

## 5.1 Proof of Lemma 2.2

The uniqueness of the solution as well as the strict concavity of $F$ are assured by the concavity of $f$ and $\eta$.

*i*) Consider the solution to the initial problem, $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ satisfying $k_t^* > 0$ and $0 < y_{t+1}^* < \eta(y_t^*)$ for any $t$. By the Euler equations, we have $f'(k_t^*) = \eta'(y_t^*)$. Since $f$ and $\eta$ are concave,

$$(k_t^*, y_t^*) = \underset{k+\theta y=z}{\operatorname{argmax}} \left(f(k) + \theta\eta(y)\right).$$

Hence for any $t$, we have $F'(z_t^*) = f'(k_t^*) = \eta'(y_t^*)$, or the sequence $\{z_t^*\}_{t=0}^{\infty}$ satisfies Euler equation:

$$u'(c_t^*) = \beta u'(c_{t+1}^*)F'(z_{t+1}^*), \quad \forall t.$$

By the transversality condition of the initial problem,

$$\begin{aligned}\lim_{t\to\infty} \beta^t u'(c_t^*) z_{t+1}^* &= \lim_{t\to\infty} \beta^t u'(c_t^*) \left(k_{t+1}^* + \theta\eta(y_{t+1}^*)\right) \\ &= 0.\end{aligned}$$

Hence the transversality condition is satisfied. The sequence $\{z_t^*\}_{t=0}^{\infty}$ solves the modified problem.

$ii)$ Consider the solution $\{\tilde{z}_t\}_{t=0}^{\infty}$ to the modified problem.

Let $(\tilde{k}_t, \tilde{y}_t) = \text{argmax}_{k+\theta y=\tilde{z}_t} \left(f(k) + \theta\eta(y)\right)$. If $\tilde{k}_t > 0$ and $0 < \tilde{y}_{t+1} < \tilde{\eta}(y_t)$ for any $t$, then $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is a feasible sequence of the initial problem.

By Lemma 2.1, we have for any $t \geq 1$, $f'(\tilde{k}_t) = \eta'(\tilde{y}_t) = F'(\tilde{z}_t)$. By the Euler equations:

$$\begin{aligned} u'(\tilde{c}_t) &= \beta u'(\tilde{c}_{t+1}) f'(\tilde{k}_{t+1}) \\ &= \beta u'(\tilde{c}_{t+1}) \eta'(\tilde{y}_{t+1}). \end{aligned}$$

Observe that for any $t \geq 1$, we have $\tilde{k}_t \leq \tilde{z}_t$ and $\tilde{y}_t \leq \dfrac{\tilde{z}_t}{\theta}$. From the transversality condition of the modified problem:

$$\begin{aligned} &\lim_{t\to\infty} \beta^t u'(\tilde{c}_t)\tilde{k}_{t+1} = 0, \\ &\lim_{t\to\infty} \beta^t u'(\tilde{c}_t)\tilde{y}_{t+1} = 0. \end{aligned}$$

The sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ satisfies the Euler equations and the transversality condition of the initial problem, hence this sequence is optimal.

## 5.2 Proof of Proposition 2.2

$i)$ From the Inada conditions, one has $f'(k^s) = \eta'(y^s) = F'(z^s) = \frac{1}{\beta}$. This implies $0 < y^s < \eta(y^s)$. Hence the sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ with $k_t^* = k^s$ and $y_t^s = y^s$ for any $t$ satisfies the Euler equations and the transversality condition of the initial problem with initial state $(k_0, y_0) = (k^s, y^s)$.

$ii)$ Take a neighborhood $\mathcal{V}_z$ of $z^s$ such that if $z_0 \in \mathcal{V}_z$, the optimal sequence $\{z_t^*\}_{t=0}^{\infty}$ is a subset of $\mathcal{V}_z$ and converges to $z^s$. Define $\tilde{\mathcal{V}}$ the set of $(k_0, y_0)$ such that $z_0 = F^{-1}\big(f(k_0 + \theta y_0)\big)$ belongs to $\mathcal{V}_z$.

Obviously, $\tilde{\mathcal{V}}$ contains a neighborhood $\mathcal{V}$ of $(k^s, y^s)$. For any $(k_0, y_0) \in \mathcal{V}$, define $z_0 = f(k_0) + \theta\eta(y_0)$. The optimal solution $\{\tilde{z}_t\}_{t=0}^{\infty}$ to the modified problem with initial $z_0$ satisfies $z_t \in \mathcal{V}_z$ for any $t$ and converges to $z^s$. Moreover, since $0 < y^s < \eta(y^s)$, the corresponding sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ satisfies $0 < \tilde{y}_{t+1} < \eta(y_t)$ for any $t$ and hence $f'(\tilde{k}_t) = \eta'(\tilde{y}_t) = F'(\tilde{z}_t)$. Obviously, this sequence satisfies the transversality condition. By Lemma 2.2, the sequence $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$ is a solution to the initial problem and from the convergence of $\{\tilde{z}_t\}_{t=0}^{\infty}$ to $z^s$, this sequence converges to $(k^s, y^s)$.

## 5.3 Proof of Lemma 2.4

The main idea of this proof is to prove that if $\Psi(k_t^*, y_t^*) \leq \Psi(k_0, y_0)$ for any $t$, then

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t u(c_t^*) &\leq \frac{u\left((1-\beta)\sum_{t=0}^{\infty}\beta^t c_t^*\right)}{1-\beta} \\ &\leq \frac{u\big(f(k_0) - k_0 + \theta(\eta(y_0) - y_0)\big)}{1-\beta} \\ &= \sum_{t=0}^{\infty} \beta^t u\big(f(k_0) - k_0 + \theta(\eta(y_0) - y_0)\big).\end{aligned}$$

This implies that the constant sequence $\{(k_0, y_0)\}_{t=0}^{\infty}$ is optimal. Hence $(k_0, y_0)$ is a steady state.

First, observe that for any $T$:

$$\begin{aligned}&\sum_{t=0}^{T} \beta^t \left(f(k_t^*) + \theta\eta(y_t^*) - k_{t+1}^* - \theta y_{t+1}^*\right) \\ &= f(k_0) + \theta\eta(y_0) + \left(\sum_{t=0}^{T-1} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*)\right) - \beta^T \left(f(k_{T+1}^*) + \theta\eta(y_{T+1}^*)\right).\end{aligned}$$

Let $T$ tend to infinity, we get

$$\sum_{t=0}^{\infty} \beta^t c_t^* = f(k_0) + \theta\eta(y_0) + \sum_{t=0}^{\infty} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*).$$

Now assume that $\Psi(k_t^*, y_t^*) \leq \Psi(k_0, y_0)$ for any $t \geq 0$. This implies

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t c_t^* &= f(k_0) + \theta\eta(y_0) + \sum_{t=0}^{\infty} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*) \\ &\leq f(k_0) + \theta\eta(y_0) + \frac{\Psi(k_0, y_0)}{1-\beta} \\ &= f(k_0) + \theta\eta(y_0) + \frac{\beta\left(f(k_0) + \theta\eta(y_0)\right) - k_0 - \theta y_0}{1-\beta} \\ &= \frac{f(k_0) - k_0 + \theta(\eta(y_0) - y_0)}{1-\beta}.\end{aligned}$$

Hence by the concavity of $u$[11]:

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) \le \frac{u\left((1-\beta)\sum_{t=0}^{\infty}\beta^t c_t^*\right)}{1-\beta} \le \frac{u\big(f(k_0)-k_0+\theta(\eta(y_0)-y_0)\big)}{1-\beta}.$$

We will prove that the hypothesis $\Psi(k_t^*, y_t^*) \le \Psi(k_0, y_0)$ for any $t \ge 0$ implies that $(k_0, y_0) \in \Gamma(k_0, y_0)$. Indeed, assume the contrary. Since $y_0 \le \eta(y_0)$, we have

$$k_0 + \theta y_0 > f(k_0) + \theta\eta(y_0),$$

otherwise $(k_0, y_0) \in \Gamma(k_0, y_0)$.

This inequality implies $k_0 > f(k_0)$. Denote by $\overline{k}$ the solution to $f(k) = k$. By the concavity of $f$, we have $\overline{k} < k_0$. Consider the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ such that $\hat{k}_0 = k_0$, $\hat{k}_t = \overline{k}$ for any $t \ge 1$, and $\hat{y}_t = y_0$ for any $t \ge 0$. This sequence is feasible. Since $\overline{k} < k_0$,

$$\begin{aligned} f(k_0) + \theta\eta(y_0) - \big(k_0 + \theta y_0\big) &< f(k_0) + \theta\eta(y_0) - \big(\overline{k} + \theta y_0\big) \\ &= f(k_0) + \theta\eta(y_0) - (\hat{k}_1 + \theta\hat{y}_1). \end{aligned}$$

For any $t \ge 1$, since $f(\overline{k}) = \overline{k}$, we have

$$\begin{aligned} f(\hat{k}_t) + \theta\eta(\hat{y}_t) - \big(\hat{k}_{t+1} + \theta\hat{y}_{t+1}\big) &= f(\overline{k}) + \theta\eta(y_0) - \big(\overline{k} + \theta y_0\big) \\ &> f(k_0) + \theta\eta(y_0) - \big(k_0 + \theta y_0\big). \end{aligned}$$

Since $\{c_t^*\}_{t=0}^{\infty}$ is the optimal consumption sequence,

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t^*) &\ge \sum_{t=0}^{\infty} \beta^t u\left(f(\hat{k}_t) + \theta\eta(\hat{y}_t) - \big(\hat{k}_{t+1} + \theta\hat{y}_{t+1}\big)\right) \\ &> \sum_{t=0}^{\infty} \beta^t u\left(f(k_0) + \theta\eta(y_0) - \big(k_0 + \theta y_0\big)\right) \\ &= \frac{u\left(f(k_0) + \theta\eta(y_0) - \big(k_0 + \theta y_0\big)\right)}{1-\beta}, \end{aligned}$$

a contradiction.

[11]Recall that for $0 < \beta < 1$, $(1-\beta)\sum_{t=0}^{\infty}\beta^t = 1$. Hence $(1-\beta)\sum_{t=0}^{\infty}\beta^t u(c_t^*) \le u\left((1-\beta)\sum_{t=0}^{\infty}\beta^t c_t^*\right)$.

Hence $(k_0, y_0) \in \Gamma(k_0, y_0)$. The sequence $\{k_t, y_t\}_{t=0}^{\infty}$ such that for any $t$, $k_t = k_0$, $y_t = y_0$ is feasible. By the choice of $\{c_t^*\}_{t=0}^{\infty}$,

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t u(c_t^*) &= \sum_{t=0}^{\infty} \beta^t u\big(f(k_t^*) - k_{t+1}^* + \theta(\eta(y_t^*) - y_{t+1}^*)\big) \\ &\geq \sum_{t=0}^{\infty} \beta^t u\big(f(k_0) - k_0 + \theta(\eta(y_0) - y_0)\big) \\ &= \frac{u\big(f(k_0) - k_0 + \theta(\eta(y_0) - y_0)\big)}{1-\beta} \\ &\geq \sum_{t=0}^{\infty} \beta^t u(c_t^*).\end{aligned}$$

The optimal path is unique, implying $k_t^* = k_0$ and $y_t^* = y_0$ for any $t \geq 0$. The couple $(k_0, y_0)$ is the steady state.

For the case where the optimal sequence is not constant, the above arguments imply the existence of $t$ such that $\Psi(k_t^*, y_t^*) > \Psi(k_0, y_0)$.

## 5.4 Proof of Proposition 2.3

The proof is divided in some intermediary steps.

i) There exists $T$ such that $y_t^* < \eta(y_t^*)$ for any $t \geq T$.

ii) The equality $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) = \Psi(k^s, y^s)$.

iii) The convergence of the optimal path.

$(i)$ First, we prove the existence of some $T$ such that $y_T^* < \eta(y_T^*)$.

Suppose the contrary, then for any $t \geq 0$ we have

$$y_{t+1}^* \leq \eta(y_t^*) \leq y_t^*.$$

The sequence $\{y_t^*\}_{t=0}^{\infty}$ is decreasing and hence converges to some $y^*$ satisfying

$$y^* \leq \eta(y^*) \leq y^*,$$

which implies that $y^* = \eta(y^*) = \overline{y}$. Recall that $\overline{y}$ is the unique solution to $\eta(y) = y$.

Now we prove the existence of some $T$ such that

$$f'(k^*_{T+1}) > \eta'(y^*_{T+1}).$$

Indeed, suppose the contrary. This implies

$$\limsup_{t\to\infty} f'(k^*_t) \leq \eta'(\overline{y}) \\ < 1.$$

By the Euler equations $u'(c^*_t) = \beta u'(c^*_{t+1}) f'(k^*_{t+1})$, there exists $T$ sufficiently big such that for any $t \geq T$, $u'(c^*_t) \leq u'(c^*_{t+1})$. By the concavity of $u$, the function $u'$ is decreasing. This implies that the truncated sequence $\{c^*_t\}_{t=T}^{\infty}$ is decreasing and converges to $c^*$.

The convergence of sequences $\{c^*_t\}_{t=T}^{\infty}$ and $\{y^*_t\}_{t=0}^{\infty}$ implies the convergence of $\{k^*_t\}_{t=0}^{\infty}$:

$$\lim_{t\to\infty} k^*_t = k^*.$$

From the Euler equations, we deduce that either $c^* = 0$, or $f'(k^*) = \frac{1}{\beta}$. The hypothesis that $f'(k^*) = \frac{1}{\beta}$, which is bigger than 1, leads us to a contradiction. Hence $c^* = 0$. Since $\lim_{t\to\infty} y^*_t = \overline{y}$, we have $\lim_{t\to\infty} k^*_t = \overline{k}$, the solution to $f(k) = k$. By the continuity of the optimal policy function, this leads to the conclusion that the consumption level at initial state $(\overline{k}, \overline{y})$ is $c^* = 0$: a contradiction.

Hence there exists some $T$ such that

$$f'(k^*_{T+1}) > \eta'(y^*_{T+1}).$$

Fix $\epsilon > 0$ sufficiently small such that:

$$f(k^*_{T+1} + \epsilon) + \theta\eta\left(y^*_{T+1} - \frac{\epsilon}{\theta}\right) > f(k^*_{T+1}) + \theta\eta\left(y^*_{T+1}\right).$$

Consider the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ defined as

$$\hat{y}_t = y_t^* \text{ for any } 0 \leq t \leq T,$$
$$\hat{y}_{T+1} = y_{T+1}^* - \frac{\epsilon}{\theta},$$
$$\hat{y}_{t+1} = y_{t+1}^* \text{ for any } t \geq T,$$
$$\hat{k}_t = k_t^* \text{ for any } 0 \leq t \leq T,$$
$$\hat{k}_{T+1} = k_{T+1}^* + \epsilon,$$
$$\hat{k}_t = k_t^* \text{ for any} t \geq T+2.$$

We can verify that the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ is feasible. We have

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*) &= \beta^{T+1}\left(u(\hat{c}_{T+1}) - u(c_{T+1}^*)\right) \\ &= \beta^{T+1} u\left(f(k_{T+1}^* + \epsilon) + \theta\eta\left(y_{T+1}^* - \frac{\epsilon}{\theta}\right) - k_{T+2}^* - \theta\eta(y_{T+2}^*)\right) \\ &- \beta^{T+1} u\left(f(k_{T+1}^*) + \theta\eta\left(y_{T+1}^*\right) - k_{T+2}^* - \theta\eta(y_{T+2}^*)\right) \\ &> 0,\end{aligned}$$

a contradiction. This contradiction comes from the hypothesis that for any $t$, $y_t^* \geq \eta(y_t^*)$.

Then there exists some $T$ such that $y_T^* < \eta(y_T^*)$. Hence $y_T^* < \overline{y}$, solution to $\eta(y) = y$. By induction, for any $t \geq T$, $y_t^* < \overline{y}$. This implies $y_t^* < \eta(y_t^*)$ for any $t \geq 0$.

$(ii)$ Consider a subsequence $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} \Psi\left(k_{t_n}^*, y_{t_n}^*\right) = \sup_{t\geq 0} \Psi(k_t^*, y_t^*).$$

Recall that $\sup_{t\geq 0} \Psi(k_t^*, y_t^*) \leq \Psi(k^s, y^s)$. Suppose that this inequality is strict.

Since the sequence $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^{\infty}$ is bounded, without loss of generality, we can assume that

$$\lim_{n\to\infty} k_{t_n}^* = k^*,$$
$$\lim_{n\to\infty} y_{t_n}^* = y^*.$$

Since $\sup_{t\geq 0} \Psi(k_t^*, y_t^*) < \Psi(k^s, y^s)$, we have $\Psi(k^*, y^*) < \Psi(k^s, y^s)$ and $(k^*, y^*)$ is not a steady state.

Let $\{\tilde{k}_t, \tilde{y}_t\}_{t=0}^{\infty}$ be the optimal path beginning from $(k^*, y^*)$. By Lemma 2.4, there exists $T$ such that

$$\Psi(\tilde{k}_T, \tilde{y}_T) > \Psi(k^*, y^*).$$

By continuity, there is a neighborhood $\mathcal{V}$ of $(k^*, y^*)$ such that for any $(k'_0, y'_0) \in \mathcal{V}$, the optimal path $\{k'_t, y'_t\}_{t=0}^{\infty}$ satisfies

$$\Psi(k'_T, y'_T) > \Psi(k^*, y^*).$$

Since the sequence $\{(k^*_{t_n}, y^*_{t_n})\}_{n=0}^{\infty}$ converges to $(k^*, y^*)$, there is $n$ sufficiently big such that $(k^*_{t_n}, y^*_{t_n}) \in \mathcal{V}$. We have

$$\begin{aligned}\Psi\left(k^*_{t_n+T}, y^*_{t_n+T}\right) &> \Psi(k^*, y^*) \\ &= \sup_{t\geq 0} \Psi(k^*_t, y^*_t),\end{aligned}$$

a contradiction. This contradiction comes from the hypothesis that $\sup_{t\geq 0} \Psi(k^*_t, y^*_t) < \Psi(k^s, y^s)$. Hence, $\sup_{t\geq 0} \Psi(k^*_t, y^*_t) = \Psi(k^s, y^s)$.

$(iii)$ For any neighborhood $\mathcal{V}$ of $(k^s, y^s)$, there is some $t$ such that $(k^*_t, y^*_t) \in \mathcal{V}$. By Proposition 2.2,

$$\lim_{t\to\infty} k^*_t = k^s \text{ and } \lim_{t\to\infty} y^*_t = y^s.$$

## 5.5 Proof of Proposition 3.2

$(i)$ Fix any $(k_0, y_0) \in S^m$. First, we prove that the constant sequence beginning from $(k_0, y_0)$ is feasible. Indeed, we only need to prove that $y_0 \leq \eta\,(y_0, E(k_0))$. Suppose the contrary, $\eta\,(y_0, E(k_0)) < y_0$. Since $\eta_1\,(0, E(k_0)) = \infty$, there exists $y$ sufficiently small such that $y < \eta\,(y, E(k_0))$. This implies $\Psi^e(y, k_0) > \Psi^e(y_0, k_0)$: a contradiction.

Consider an optimal path $\{(k^*_t, y^*_t)\}_{t=0}^{\infty}$ beginning from $(k_0, y_0)$. By the choice of $(k_0, y_0)$, for any $t$ we have $\Psi^e(k^*_t, y^*_t) \leq \Psi^e(k_0, y_0)$. Using the same arguments as in the proof of Proposition 2.3, we have

$$\sum_{t=0}^{\infty} \beta^t u\left(f(k_0) + \theta\eta\,(y_0, E(k_0)) - k_0 - \theta y_0\right) \geq \sum_{t=0}^{\infty} \beta^t u(c^*_t),$$

which implies that the constant sequence $\{(k_0, y_0)\}_{t=0}^{\infty}$ is also an optimal path beginning from $(k_0, y_0)$. Hence $(k_0, y_0)$ is a steady state of the economy.

$(ii)$ We follow the same line of arguments of the proof of Proposition 2.3. Fix $(k_0, y_0)$ and an optimal path $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ beginning from $(k_0, y_0)$. We have

$$(1-\beta)\sum_{t=0}^{\infty}\beta^t c_t^* = f(k_0) + \theta\eta\left(y_0, E(k_0)\right) + \sum_{t=0}^{\infty}\beta^t\Psi^e(k_t^*, y_t^*).$$

Assume that $\Psi^e(k_t^*, y_t^*) \leq \Psi^e(k_0, y_0)$ for any $t \geq 0$. By the concavity of $u$, one has

$$(1-\beta)\sum_{t=0}^{\infty}\beta^t u(c_t^*) \leq u\left((1-\beta)\sum_{t=0}^{\infty}\beta^t c_t^*\right).$$

This is equivalent to

$$\sum_{t=0}^{\infty}\beta^t u(c_t^*) \leq \frac{u\left(f(k_0) + \theta\eta\left(y_0, E(k_0)\right) - k_0 - \theta y_0\right)}{1-\beta}.$$

We prove that $(k_0, y_0) \in \Gamma(k_0, y_0)$. In the contrary case, this implies $k_0 > f(k_0)$. Hence $k_0 > \overline{k}$, the solution to the equation $f(k) = k$. The sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ with $\hat{k}_t = \overline{k}$ and $\hat{y}_t = y_0$ for any $t \geq 1$ is feasible.

Observe that $E(\overline{k}) < E(k_0)$. We have

$$\begin{aligned}
f(\hat{k}_0) + \theta\eta\left(\hat{y}_0, E(\hat{k}_0)\right)\Big) - \left(\hat{k}_1 + \theta\hat{y}_1\right) &= f(\overline{k}) + \theta\eta\left(\overline{y}, E(\overline{k})\right) - \left(\overline{k} + \theta\overline{y}\right) \\
&> f(k_0) + \theta\eta\left(y_0, E(k_0)\right) - \big(k_0 + \theta y_0\big).
\end{aligned}$$

For any $t \geq 1$, since $f(\overline{k}) = \overline{k}$, we have

$$\begin{aligned}
f(\hat{k}_t) + \theta\eta\left(\hat{y}_t, E(\hat{k}_t)\right)\Big) - \left(\hat{k}_{t+1} + \theta\hat{y}_{t+1}\right) &= f(\overline{k}) + \theta\eta\left(y_0, E(l_0)\right) - \left(\overline{k} + \theta y_0\right) \\
&> f(k_0) + \theta\eta\left(y_0, E(k_0)\right) - \big(k_0 + \theta y_0\big).
\end{aligned}$$

Since $\{c_t^*\}_{t=0}^{\infty}$ is an optimal consumption sequence, this implies

$$\begin{aligned}
\sum_{t=0}^{\infty}\beta^t u(c_t^*) &\geq \sum_{t=0}^{\infty}\beta^t u\left(f(\hat{k}_t) + \theta\eta\big(\hat{y}_t, E(\hat{k}_t)\big)\right) \\
&> \sum_{t=0}^{\infty}\beta^t u\left(f(k_0) + \theta\eta\left(y_0, E(k_0)\right) - \big(k_0 + \theta y_0\big)\right) \\
&= \frac{u\left(f(k_0) + \theta\eta\left(y_0, E(k_0)\right) - \big(k_0 + \theta y_0\big)\right)}{1-\beta},
\end{aligned}$$

a contradiction.

Hence $(k_0, y_0) \in \Gamma(k_0, y_0)$, implying that the sequences $\{c_t^*\}_{t=0}^{\infty}$ and $\{\Psi^e(k_t^*, y_t^*)\}_{t=0}^{\infty}$ are constant. Hence for any $t$,

$$f(k_t^*) + \theta\eta\left(y_t^*, E(k_t^*)\right) - k_{t+1}^* - \theta y_{t+1}^* = f(k_0) + \theta\eta\left(y_0, E(k_0)\right) - k_1^* - \theta y_1^*,$$

$$\beta\big(f(k_t^*) + \theta\eta\left(y_t^*, E(k_t^*)\right)\big) - k_t^* - \theta y_t^* = \beta\big(f(k_0) + \theta\eta\left(y_0, E(k_0)\right)\big) - k_0^* - \theta y_0^*.$$

We have

$$k_t^* + \theta y_t^* = \beta(k_{t+1}^* + \theta y_{t+1}^*) + \big(-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)\big).$$

Invoking the same argument for $t+1$, we obtain

$$k_{t+1}^* + \theta y_{t+1}^* = \beta(k_{t+2}^* + \theta y_{t+2}^*) + \big(-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)\big).$$

This implies

$$\begin{aligned} k_t^* + \theta y_t^* &= \beta(k_{t+1}^* + \theta y_{t+1}^*) + \big(-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)\big) \\ &= \beta^2(k_{t+2}^* + \theta y_{t+2}^*) + \beta\big(-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)\big) + \big(-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)\big) \\ &= \cdots \\ &= \beta^T(k_{t+T}^* + \theta y_{t+T}^*) + \big(-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)\big) \sum_{s=0}^{T-1} \beta^s. \end{aligned}$$

Let $T$ tend to infinity, we get for any $t$,

$$k_t^* + \theta y_t^* = \frac{-\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0)}{1-\beta}.$$

Let $t = 1$. This implies $k_1^* + \theta y_1^* = k_0 + \theta y_0$. Hence for any $t \geq 0$ we have

$$k_t^* + \theta y_t^* = k_0 + \theta y_0.$$

Since the consumption sequence is constant, we get for any $t \geq 0$,

$$f(k_t^*) + \theta\eta\left(y_t^*, E(k_t^*)\right) = f(k_0) + \eta\left(y_0, E(k_0)\right).$$

These two equalities imply that $(k_0, y_0)$ is a steady state.

The conclusion that $(k_0, y_0)$ belongs to the set of steady states comes from the hypothesis that $\Psi^e(k_t^*, y_y^*) \leq \Psi^e(k_0, y_0)$ for any $t \geq 0$. Therefore if $(k_0, y_0)$ is not a steady state, there exists $t$ such that

$$\Psi^e(k_t^*, y_t^*) > \Psi^e(k_0, y_0).$$

## 5.6 Proof of Lemma 3.1

By Assumption **H3**, the uniqueness of the steady state is ensured. For each $0 \le z' \le G(z)$, define $V(z, z') = u\left(G(z) - z'\right)$. The function $V$ is the indirect utility function of the modified economy.

By the concavity of $u$ and the monotonicity of $G$, the indirect utility function $V$ has increasing differences (see Amir [1]). Every optimal path of the modified problem is hence monotonic.

We will prove the following claim: for any initial state $z_0 > 0$, every optimal path beginning from $z_0$ converges monotonically to $z^s$. Precisely, let $\{z_t^*\}_{t=0}^{\infty}$ be an optimal path beginning from $z_0$. If $z_0 \le z^s$ then this path is increasing and converges to $z^s$. Otherwise, if $z_0 \ge z^s$, this path is decreasing and converges to $z^s$.

Indeed, consider the case $0 < z_0 < z^s$. Assume that the sequence $\{z_t^*\}_{t=0}^{\infty}$ is strictly decreasing. For fixed $z < z^s$, consider the following function with variable $z'$ belonging to $[0, z]$:

$$w(z') = u\left(G(z) - z'\right) + \frac{\beta}{1-\beta} u\left(G(z') - z'\right).$$

By the concavity of $u$,

$$\begin{aligned} w'(z') &= -u'\left(G(z) - z'\right) + \frac{\beta}{1-\beta} u'\left(G(z') - z'\right)\left(G'(z') - 1\right) \\ &\ge -u'\left(G(z') - z'\right) + \frac{\beta}{1-\beta} u'\left(G(z') - z'\right)\left(G'(z') - 1\right) \\ &= u'\left(G(z') - z'\right) \times \frac{\beta G'(z') - 1}{1-\beta} \\ &> 0. \end{aligned}$$

This implies that the function $w$ is strictly increasing in $[0, z]$. Hence,

$$\begin{aligned} \frac{u\left(G(z) - z\right)}{1-\beta} &= w(z) \\ &\ge w(z') \\ &= u\left(G(z) - z'\right) + \frac{\beta}{1-\beta} u\left(G(z') - z'\right), \end{aligned}$$

for any $0 \le z' \le z$.

The hypothesis that $\{z_t^*\}_{t=0}^{\infty}$ is decreasing implies

$$\begin{aligned}\frac{u\left(G(z_0)-z_0\right)}{1-\beta} &\geq u\left(G(z_0)-z_1^*\right)+\beta\frac{u\left(G(z_1^*)-z_1\right)}{1-\beta}\\ &\geq u\left(G(z_0)-z_1^*\right)+\beta u\left(G(z_1^*)-z_2^*\right)+\beta^2\frac{u\left(G(z_2^*)-z_2^*\right)}{1-\beta}\\ &\dots\\ &\geq \sum_{t=0}^{T}\beta^t u\left(G(z_t^*)-z_{t+1}^*\right)+\beta^{T+1}\frac{u\left(G(z_{T+1}^*)-z_{T+1}^*\right)}{1-\beta}\\ &\dots\\ &\geq \sum_{t=0}^{\infty}\beta^t u\left(G(z_t^*)-z_{t+1}^*\right).\end{aligned}$$

Hence $(z_0, z_0, \dots)$ is also an optimal path, which implies $z_0 = z^s$: a contradiction.

Hence the sequence $\{z_t^*\}_{t=0}^{\infty}$ is increasing and converges to $z^s$. For $z_0 > z^s$, using the same arguments, we can prove that any optimal path beginning from $z_0$ is decreasing and converges to $z^s$.

## 5.7 Proof of Proposition 3.3

$(i)$ The proof follows the same arguments as in Section 2. We know that for any $z_0$, the optimal path of the modified problem converges monotonically to the steady state $z^s$. We have $z^s = k^s + \theta y^s$.

Consider an optimal path of the modified problem $\{z_t^*\}_{t=0}^{\infty}$. Define $(k_t^*, y_t^*)$ by

$$(k_t^*, y_y^*) = \underset{k+\theta y = z_t^*}{\operatorname{argmax}}\big[f(k) + \theta\eta(y, \alpha k)\big].$$

Since $y^s < g(y^s)h(\alpha k^s)$, for $(k_0, y_0)$ belonging to a neighborhood of $(k^s, y^s)$, the corresponding sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ satisfied $y_{t+1}^* < g(y_t^*)h(\alpha k_t^*)$ for any $t \geq 0$. This implies the sequence $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$ is feasible and hence is an optimal path of the initial problem. This sequence converges to $(k^s, y^s)$.

$(ii)$ Fix $(k_0, y_0)$ and an optimal path $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$. Take the subsequence $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} \Psi^e(k_{t_n}^*, y_{t_n}^*) = \sup_{t\geq 0} \Psi^e(k_t^*, y_t^*).$$

Invoking the same arguments as in the proof of Proposition 2.3, the sequence $\{(k^*_{t_n}, y^*_{t_n})\}_{n=0}^{\infty}$ converges to a steady state. By the uniqueness of the steady state, we have

$$\lim_{n\to\infty}(k^*_{t_n}, y^*_{t_n}) = (k^s, y^s).$$

By part $(i)$, this implies that for some $n$ sufficiently big, the point $(k^*_{t_n}, y^*_{t_n})$ belongs to the neighborhood $\mathcal{V}$ of $(k^s, y^s)$ and there exists an optimal path $\{(k'_{t_n+t}, y'_{t_n+t})\}_{t=0}^{\infty}$ beginning from $(k^*_{t_n}, y^*_{t_n})$ which converges to $(k^s, y^s)$. Define the sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ as

$$(\hat{k}_t, \hat{y}_t) = \begin{cases} (k^*_t, y^*_t) \text{ for } 0 \le t \le t_n, \\ (k'_t, y'_t) \text{ for } t \ge t_n. \end{cases}$$

The sequence $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$ is an optimal path beginning from $(k_0, y_0)$ converging to $(k^s, y^s)$.

## 5.8 Proof of Proposition 3.4

Since $G(z) < z$ for any $z > z^m$, we only have to consider $z \in [0, z^m]$. We prove that the function $G$ is strictly concave in $[0, z^m]$, hence the solution to $G'(z) = \frac{1}{\beta}$ is unique, and Assumption **H3** is satisfied.

Precisely,

i) For each $z$, there exists unique $(k(z), y(z))$ which maximizes $f(k)+\theta g(y)h(\alpha k)$ under constraint $k + \theta y \le z$.

ii) The function $k(z)$ is increasing in respect to $z$.

iii) The function $G$ is strictly concave and there exists a unique steady $z^s$, which is the solution to $G'(z) = \frac{1}{\beta}$.

$(i)$ For $z \ge 0$, we must find $k$ which maximizes

$$\zeta(k) = f(k) + \theta g\left(\frac{z-k}{\theta}\right) h(\alpha k).$$

We have

$$\zeta''(k) = f''(k) + \frac{1}{\theta} g''\left(\frac{z-k}{\theta}\right) h(\alpha k) - 2\alpha g'\left(\frac{z-k}{\theta}\right) h'(\alpha k) + \alpha^2 \theta g\left(\frac{z-k}{\theta}\right) h''(\alpha k).$$

Assumption **H5** implies that $\zeta$ is strictly concave. Hence there exists unique $k(z) \in [0, z]$ maximizing $\zeta(k)$.

$(ii)$ It is easy to verify that $0 < k(z) < z$ for $z > 0$. The value $k(z)$ is hence a solution to

$$f'(k) - g'\left(\frac{z-k}{\theta}\right) h(\alpha k) + \theta \alpha g\left(\frac{z-k}{\theta}\right) h'(\alpha k) = 0.$$

By the Implicit Function Theorem, we get

$$\begin{aligned} k'(z) &= -\frac{-\frac{1}{\theta} g''\left(\frac{z-k}{\theta}\right) h(\alpha k) + \alpha g'\left(\frac{z-k}{\theta}\right) h'(\alpha k)}{f''(k) + \frac{1}{\theta} g''\left(\frac{z-k}{\theta}\right) h(\alpha k) - 2\alpha g'\left(\frac{z-k}{\theta}\right) h'(\alpha k) + \alpha^2 \theta g\left(\frac{z-k}{\theta}\right) h''(\alpha k)} \\ &> 0, \end{aligned}$$

since the nominator is positive and the denominator is negative.

$(iii)$ For any $z > 0$,

$$\begin{aligned} G'(z) &= f'(k(z))k'(z) + g'\left(\frac{z-k(z)}{\theta}\right)(1-k'(z))h(\alpha k(z)) + \alpha g\left(\frac{z-k(z)}{\theta}\right) h'(\alpha k(z))k'(z) \\ &= g'\left(\frac{z-k(z)}{\theta}\right) h(\alpha k(z)). \end{aligned}$$

This implies

$$\begin{aligned} G''(z) &= \frac{1}{\theta} g''\left(\frac{z-k(z)}{\theta}\right)(1-k'(z))h(\alpha k(z)) + \alpha g'\left(\frac{z-k(z)}{\theta}\right) h'(\alpha k(z))k'(z) \\ &= \frac{1}{\theta} g''\left(\frac{z-k(z)}{\theta}\right) h(\alpha k(z)) \\ &\quad + k'(z)\left(-\frac{1}{\theta} g''\left(\frac{z-k(z)}{\theta}\right) h(\alpha k(z)) + \alpha g'\left(\frac{z-k(z)}{\theta}\right) h'(\alpha k(z))\right) \\ &< 0, \end{aligned}$$

since the two terms are negative. The function $G$ is strictly concave.

## 5.9 Proof of Corollary 3.1

Since $f''(k) \leq 0$ for any $k$, the condition $(i)$ in Assumption **H5** is satisfied. Moreover, since $\alpha g'\left(\frac{z-k}{\theta}\right) h'(\alpha k)$ and $\alpha^2 g\left(\frac{z-k}{\theta}\right)$ are positive, the assumption in the statement of this corollary implies the satisfaction of condition $(ii)$ in Assumption **H5**. Assumption **H5** is hence satisfied. Invoking Proposition 3.3 completes the proof.

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